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Bicomplexes and Bäcklund transformations

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Abstract

A bicomplex is a simple mathematical structure, in particular associated with completely integrable models. The conditions defining a bicomplex are a special form of a parameter-dependent zero-curvature condition. We generalize the concept of a Darboux matrix to bicomplexes and use it to derive Bäcklund transformations for several models. The method also works for Moyal-deformed equations with a corresponding deformed bicomplex.

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1. Introduction

Bäcklund transformations arose in the 19th century in a differential geometric context [1]. Despite the fact that this is already a rather old subject, it is still the subject of several recent articles. Indeed, in the case of many nonlinear equations, a Bäcklund transformation (BT) remains the only hope to construct sufficiently complicated exact solutions.

The essence of the concept of a BT is basically the following (see [2–5], for example). Suppose we have two (systems of) partial differential equations[3] $EQ_1[u_1] = 0$ and $EQ_2[u_2] = 0$, depending on a variable u_1 and its partial derivatives, respectively u_2 and its partial derivatives. A BT is then given by relations between the two variables and their partial derivatives which determine u_2 in terms of u_1 such that $EQ_2[u_2] = 0$ if $EQ_1[u_1] = 0$ holds. So, given a solution of $EQ_1 = 0$, it determines a corresponding solution of $EQ_2 = 0$. This will only be of help, of course, if the relations between u_1 and u_2 are considerably simpler than the equations $EQ_1 = 0$ and $EQ_2 = 0$, and if some solutions of one of these equations are known. For instance, $EQ_1 = 0$ and $EQ_2 = 0$ may be higher-order partial differential equations and the BT only of first order. If $EQ_1 = EQ_2$, then such a transformation is called an *auto-Bäcklund transformation*. It can be used to generate new solutions of $EQ_1 = 0$ from given ones. How to find (useful) BTs? For most completely integrable models, ways to

[3] We may consider difference equations as well.

construct BTs are known. The existence of such a transformation is usually taken as a criterion for complete integrability.

A large class of completely integrable equations in two (spacetime) dimensions admits a *zero-curvature formulation*

$$U_t - V_x + [U, V] = 0 \tag{1.1}$$

(see [6], in particular) with matrices $U(u, \lambda)$ and $V(u, \lambda)$ depending on a parameter λ, besides the dependent variable u. Here, U_t and V_x denote the partial derivatives of U and V with respect to the coordinates t and x, respectively. Equation (1.1) is the compatibility condition of the linear system

$$z_x = U(u, \lambda)\, z \qquad z_t = V(u, \lambda)\, z \tag{1.2}$$

with a (matrix) 'wavefunction' z. A special class of auto-BTs is then obtained via λ-dependent transformations

$$z' = Q(u, u', \lambda)\, z \tag{1.3}$$

depending on two fields, u and u', which preserve the form of the linear system, so that

$$z'_x = U(u', \lambda)\, z' \qquad z'_t = V(u', \lambda)\, z'. \tag{1.4}$$

Then, if u is a solution of the equation modelled by the zero-curvature condition, u' is also a solution. Q has to satisfy the equations

$$Q_x = U(u')\, Q - Q\, U(u) \qquad Q_t = V(u')\, Q - Q\, V(u) \tag{1.5}$$

and is called a *Darboux matrix* [7,8][4]. It is not known, except for special examples, whether all auto-BTs of an integrable model, possessing a zero-curvature formulation, can be recovered in this way. Many auto-BTs are known to be of this Darboux form, however (see [7, 11], for example). In many cases, an ansatz for Q which is linear in λ suffices [8]. The important 'dressing method' of Zakharov and Shabat [12] actually involves the construction of a Darboux matrix.

In terms of the λ-dependent 'covariant derivative' $D = d - U\,dx - V\,dt = d + A$, where d is the exterior derivative on $\mathbb{R}^2$ (with coordinates x and t), (1.2) and (1.4) can be written as $Dz = 0$ and $D'z' = 0$, respectively, and (1.1) becomes $D^2 = 0$, which is[5] $F = dA + A \wedge A = 0$. Equation (1.5) can be rewritten as

$$A'Q = Q\,A - dQ \tag{1.6}$$

which, for an invertible Q, is the transformation law of a connection ('gauge potential') under a gauge transformation given by Q. This is equivalent to the covariance property

$$D'Q = QD \tag{1.7}$$

of the covariant derivative. This scheme can be generalized to hetero-BTs where Q relates two different zero-curvature equations[6].

In the special case where D depends linearly on λ, it naturally splits into two linear operators which do not depend on λ and we have an example of a bicomplex [14] (and even a bi-differential calculus [15]). A *bicomplex* is an $\mathbb{N}_0$-graded linear space (over $\mathbb{R}$) $M = \bigoplus_{s\geqslant 0} M^s$ together with two linear maps $\mathcal{D}, \mathrm{D} : M^s \to M^{s+1}$ satisfying

$$\mathcal{D}^2 = 0 \qquad \mathrm{D}^2 = 0 \qquad \mathrm{D}\,\mathcal{D} + \mathcal{D}\,\mathrm{D} = 0. \tag{1.8}$$

[4] This is sometimes called a *Darboux transformation* (see [9], for example), but should not be confused with the classical Darboux transformation [10], which, however, is indeed related to a Darboux matrix [8].

[5] Without further conditions, $F = 0$ is simply solved by $A = g^{-1}\,\mathrm{d}g$ with an invertible matrix g. If we require the dependence of A on λ to be polynomial or rational, this results in nontrivial equations, however.

[6] Various relations between zero-curvature conditions and BTs have also been discussed in [2–4, 13], for example.

Introducing an auxiliary real parameter λ, these equations can be written as a generalized parameter-dependent zero-curvature condition[7],

$$(\mathcal{D} - \lambda\,\mathrm{D})^2 = 0. \tag{1.9}$$

Typically, the maps $\mathcal{D}$ and D depend on a (set of) variable(s) u such that the bicomplex conditions hold if and only if u is a solution of a system of (e.g. partial differential) equations. The corresponding linear system is

$$(\mathcal{D} - \lambda\,\mathrm{D})\,z = 0 \tag{1.10}$$

(for $z \in M^0$), or a slight modification of it. The simple linear λ-dependence of this linear system has to be contrasted with the, in general, nonlinear λ-dependence of U and V in (1.2). Thus, at first sight, the bicomplex formulation looks like a severely restricted zero-curvature condition[8]. However, rewriting the bicomplex equations, if possible, in the form (1.1) and (1.2), in general results in a nonlinear λ-dependence of the corresponding U and V, but in general it will not be possible to rewrite a given zero-curvature formulation, showing a nonlinear λ-dependence, in bicomplex form[9], although it is possible that a bicomplex formulation exists for this model. Disregarding the λ-dependence, the structure of the bicomplex equations is somewhat less restrictive than (1.1). In a series of papers [14–20], several known integrable models have been cast into the bicomplex form, some new models have been constructed and it has been demonstrated, in particular, how conservation laws can be derived from it in the case of evolution-type equations. A bridge from bi-Hamiltonian systems to bicomplexes has been established in [21]. It should be stressed, however, that a bicomplex structure is much more general and does not presuppose the existence of a symplectic or Hamiltonian structure.

The idea of a Darboux matrix is immediately carried over to bicomplexes. Let $\mathcal{B}_i = (M_i, \mathcal{D}_i, \mathrm{D}_i)$, $i = 1, 2$, be two bicomplexes depending on variables u_1 and u_2, respectively. We propose the following definition.

Definition. *A Darboux–Bäcklund transformation (DBT) for the two bicomplexes $\mathcal{B}_1$ and $\mathcal{B}_2$ is given by a λ-dependent linear operator $Q(\lambda) : M_1 \to M_2$ such that*[10]

$$(\mathcal{D}_2 - \lambda\,\mathrm{D}_2)\,Q(\lambda) = Q(\lambda)\,(\mathcal{D}_1 - \lambda\,\mathrm{D}_1) \tag{1.11}$$

for all λ. This is an auto-Darboux–Bäcklund transformation if the two bicomplexes are associated with the same equation[11].

Since the bicomplex maps depend on the fields u_1 and u_2 (which are solutions of the respective field equations), so does Q as a consequence of (1.11). This implies

$$Q(\lambda)\,(\mathcal{D}_1 - \lambda\,\mathrm{D}_1)^2 = (\mathcal{D}_2 - \lambda\,\mathrm{D}_2)^2\,Q(\lambda). \tag{1.12}$$

Hence, if u_1 is a solution of the equation associated with $\mathcal{B}_1$, then $(\mathcal{D}_2 - \lambda\,\mathrm{D}_2)^2\,Q(\lambda) = 0$. If $Q(\lambda)$ is invertible, an assumption which we shall make in all our examples, this implies that u_2 is a solution of the equation associated with $\mathcal{B}_2$. If $Q(\lambda)$ is not invertible, this cannot be concluded, in general.

[7] There are examples in section 4 where the bicomplex maps $\mathcal{D}$ and D involve complex quantities but are *not* linear over $\mathbb{C}$. The bicomplex equations can then only be cast into this form for *real* λ.

[8] Actually many integrable models can be derived by reduction of the self-dual Yang–Mills equation, which possesses a zero-curvature formulation linear in λ.

[9] Clearly, this applies in particular to zero-curvature equations with a nonrational dependence on λ (see [6], for example).

[10] The requirement of 'form invariance' of U and V in the Darboux matrix formalism is here replaced by the requirement that Q preserves the linear λ-dependence of the operator $\mathcal{D} - \lambda\,\mathrm{D}$.

[11] If Q does not depend on λ and is invertible, an auto-DBT reduces to an equivalence transformation of the two bicomplexes.

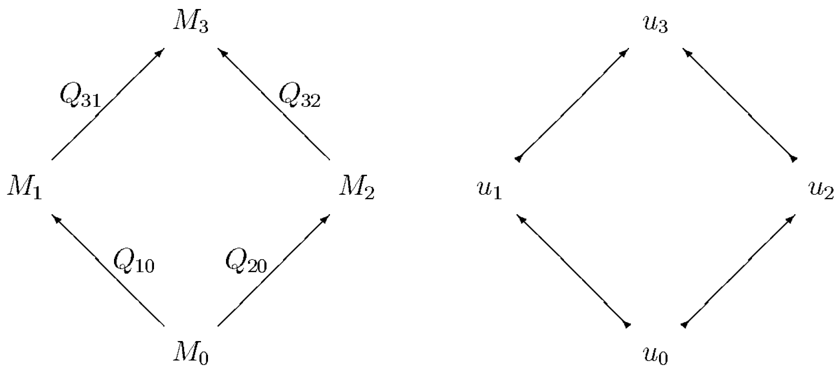

Figure 1. A diagram of DBTs and a corresponding induced mapping of solutions in the case of commutativity of the first diagram.

Given three bicomplexes which are connected by DBTs $Q_{21} : M_1 \to M_2$ and $Q_{32} : M_2 \to M_3$, the composition $Q_{32}\, Q_{21}$ is also a DBT:

$$(\mathcal{D}_3 - \lambda\, \mathrm{D}_3)\, Q_{32}(\lambda)\, Q_{21}(\lambda) = Q_{32}(\lambda)\, Q_{21}(\lambda)\, (\mathcal{D}_1 - \lambda\, \mathrm{D}_1). \tag{1.13}$$

Let us now consider four bicomplexes $\mathcal{B}_i$, $i = 0, 1, 2, 3$, with DBTs $Q_{10}, Q_{20}, Q_{31}, Q_{32}$. Suppose we can solve the respective DBT conditions such that solutions u_i of $EQ_i = 0$ can be expressed via the DBT Q_{ij} in terms of solutions of $EQ_j = 0$. The condition

$$Q_{31}\, Q_{10} = Q_{32}\, Q_{20} \tag{1.14}$$

is then in many cases strong enough to guarantee that a solution u_0 of $EQ_0 = 0$ is taken via $Q_{31}\, Q_{10}$ and also via $Q_{32}\, Q_{20}$ to the same solution u_3 of $EQ_3 = 0$ (see figure 1). The condition (1.14) is our formulation of the 'permutability theorem' (see [1,3], for example). In the case of auto-DBTs, it leads to nonlinear superposition rules for solutions of the respective equation. Typically, a BT (obtained from a DBT) depends on some arbitrary constants. The condition (1.14) then enforces relations between the corresponding constants on the left ($u_0 \mapsto u_1 \mapsto u_3$) and the right way ($u_0 \mapsto u_2 \mapsto u_3$) in the right-hand diagram of figure 1. The usual formulation of a permutability theorem is that if these relations between the constants hold, then the two ways yield the same solution u_3.

The linear dependence on the parameter λ in the bicomplex zero-curvature formulation greatly simplifies the derivation of DBTs. Specializing to various models, one easily recovers well known BTs. For this purpose we make an ansatz

$$Q(\lambda) = \sum_{k=0}^{N} \lambda^k Q^{(k)} \tag{1.15}$$

with some $N \in \mathbb{N}$ and $Q^{(k)}$ not depending on λ. The DBT condition (1.11) then splits into the system of equations

$$\begin{aligned} &\mathcal{D}_2\, Q^{(0)} - Q^{(0)}\, \mathcal{D}_1 = 0 \\ &\mathcal{D}_2\, Q^{(k)} - Q^{(k)}\, \mathcal{D}_1 = \mathrm{D}_2\, Q^{(k-1)} - Q^{(k-1)}\, \mathrm{D}_1 \qquad (k = 1, \ldots, N) \\ &\mathrm{D}_2\, Q^{(N)} - Q^{(N)}\, \mathrm{D}_1 = 0. \end{aligned} \tag{1.16}$$

We speak of a *primary DBT* when $N = 1$, of a *secondary DBT* when $N = 2$ and so forth. The composition of N primary DBTs is obviously at most an N-ary DBT. Generically, it will be indeed an N-ary DBT[12]. There may be higher DBTs which are not obtained in this way, however.

If two bicomplexes admit an *invertible* DBT, there is an equivalent DBT-problem with $Q(\lambda)$ acting on a single bicomplex space M and $Q^{(0)} = I$, the identity operator[13]. Expressing the dependence of $\mathcal{D}_i$ on a solution u_i explicitly as $\mathcal{D}_i[u_i]$, the first of equations (1.16) then requires $\mathcal{D}_1[u_1] = \mathcal{D}_2[u_2]$ for all solutions u_1 and u_2 related via $Q(\lambda)$. Let us consider the special case where the equation under consideration admits a bicomplex $\mathcal{B}_1$ such that $\mathcal{D}_1$ does not depend on u. In this case we write $\mathcal{D}_1 = \delta$ and obtain $\mathcal{D}_2 = \delta$. The DBT-problem then takes the form $(\delta - \lambda\,\mathrm{D}_2[u_2])\,Q(\lambda) = Q(\lambda)\,(\delta - \lambda\,\mathrm{D}_1[u_1])$. Moreover, if we look for *auto*-DBTs, the bicomplexes $\mathcal{B}_1 = (M, \delta, \mathrm{D}_1)$ and $\mathcal{B}_2 = (M, \delta, \mathrm{D}_2)$ have to be equivalent. This means that the respective sets of bicomplex equations, which depend on u, must both be satisfied if u solves the equation for which $\mathcal{B}_1$ and $\mathcal{B}_2$ are bicomplexes. If this holds for $\mathcal{B}_1$, then an obvious way to achieve that it also holds for $\mathcal{B}_2$ is to choose $\mathrm{D}_2[u] = \mathrm{D}_1[u] = \mathrm{D}[u]$. Since δ is common to both bicomplexes, we should actually hardly expect $\mathrm{D}_1[u]$ and $\mathrm{D}_2[u]$ to differ in a nontrivial way, although exceptions may exist[14]. This motivates us to restrict the invertible auto-DBT condition for an equation which possesses a bicomplex $\mathcal{B} = (M, \delta, \mathrm{D})$, where δ does not depend on u, to the form

$$(\delta - \lambda\,\mathrm{D}[u_2])\,Q(\lambda) = Q(\lambda)\,(\delta - \lambda\,\mathrm{D}[u_1]) \tag{1.17}$$

with $Q^{(0)} = I$. Then we are dealing with a *single* bicomplex only. This restricted form of the auto-DBT condition underlies all examples in section 4.

How severe is the above restriction on the bicomplex $\mathcal{B}_1$ (and thus $\mathcal{B}$)? Splitting a given bicomplex map $\mathcal{D}$ as $\mathcal{D} = \delta + B$ with a suitable operator δ which is independent of a solution u and satisfies $\delta^2 = 0$, the generalized curvature $\mathcal{F} = [\delta, B] + B^2$ vanishes (see also section 3). $\mathcal{F}$ generalizes the classical differential geometric formula for the curvature of a connection one-form B if δ is given by an exterior derivative on some manifold. In that case, it is well known that a gauge transformation exists which transforms B to $B' = 0$ so that $\mathcal{D}' = \delta$. In the much more general bicomplex framework we do not have a corresponding theorem at hand, though an analogous result should be expected for relevant classes of bicomplexes (see section 5 for an example). This would mean that, if a bicomplex exists, then also a bicomplex exists with a map $\mathcal{D}$ which is independent of the solution of the underlying equation. At least, this motivates a corresponding ansatz. In practice, however, it is often difficult enough to find *any* bicomplex for some equation and it is then hardly possible to find a concrete transformation to such a special bicomplex. Moreover, such a transformation may change the concrete form of the equation (cf section 5), which then possibly makes it difficult to apply corresponding results (e.g. concerning DBTs) to the original problem[15]. Furthermore, one should keep in mind that interesting examples may exist for which the above special bicomplex form cannot be reached. Of course, in such a case the DBT method can still be applied, although the calculations will be more involved, in general.

In [17–19] we constructed bicomplexes for various Moyal-deformed classical integrable

[12] An exception appears in the Liouville example treated in section 2, where the composition of primary DBTs of the form (2.12) is again a primary DBT.

[13] If $Q(\lambda)$ is invertible, then $Q^{(0)}$ determines an equivalence transformation of bicomplexes. Introducing $\mathcal{D}'_2 = (Q^{(0)})^{-1}\,\mathcal{D}_2\,Q^{(0)}$ and $\mathrm{D}'_2 = (Q^{(0)})^{-1}\,\mathrm{D}_2\,Q^{(0)}$, we have $\tilde{Q}(\lambda)\,(\mathcal{D}_1 - \lambda\,\mathrm{D}_1) = (\mathcal{D}'_2 - \lambda\,\mathrm{D}'_2)\,\tilde{Q}(\lambda)$ with $\tilde{Q}(\lambda) = (Q^{(0)})^{-1}\,Q(\lambda) : M_1 \to M_1$.

[14] If $u \mapsto \tilde{u}$ is a symmetry transformation of the equation for u, then we may also choose $\mathrm{D}_2[u] = \mathrm{D}_1[\tilde{u}]$. Note that a symmetry of the equation is not a symmetry (equivalence transformation) of an associated bicomplex, in general. The consequences of this observation have still to be explored.

[15] In particular, the nonlinear Schrödinger equation (see section 4.2.7) is known to be 'gauge equivalent' to the Heisenberg magnet model (see [6], for example). This can be understood as an equivalence of bicomplexes [14, 19].

models. Here, the ordinary product of functions is replaced by an associative, noncommutative $*$-product [22]. The definition of a bicomplex, and in particular (1.8), as well as our definition of a bicomplex DBT, still applies. Also in this case a DBT provides us with a helpful solution generating technique, as will be demonstrated with an example in section 4.

Section 2 treats the example of the Liouville equation and its discretization. In section 3 we elaborate our definition of a bicomplex DBT for a 'dressed' form of the bicomplex maps [14,15] and the restricted case where all maps act on the same bicomplex space M. Section 4 then shows how to recover auto-BTs for several well known integrable models from a bicomplex DBT. Section 5 deals with the Harry Dym (HD) equation (see [23], in particular) and section 6 collects some conclusions.

2. Example. Liouville bicomplex

In many examples, the bicomplex space can be chosen as $M = M^0 \otimes \Lambda_n$ where $\Lambda_n = \bigoplus_{r=0}^n \Lambda^r$ is the exterior algebra of an n-dimensional real vector space with a basis ξ^r, $r = 1, \ldots, n$, of Λ^1. It is then sufficient to define the bicomplex maps $\mathcal{D}$ and D on M^0 since via

$$\mathrm{D}\Bigg(\sum_{i_1,\ldots,i_r=1}^n \phi_{i_1\ldots i_r}\, \xi^{i_1} \cdots \xi^{i_r} \Bigg) = \sum_{i_1,\ldots,i_r=1}^n (\mathrm{D}\phi_{i_1\ldots i_r})\, \xi^{i_1} \cdots \xi^{i_r} \tag{2.1}$$

(and correspondingly for $\mathcal{D}$) they extend as linear maps to the whole of M[16]. In the case of Λ_2, we denote the two basis elements of Λ^1 as τ, ξ. They satisfy $\xi^2 = 0 = \tau^2$ and $\xi\, \tau = -\tau\, \xi$.

Liouville equation. Let $M = C^\infty(\mathbb{R}^2, \mathbb{R}^2) \otimes \Lambda_2$. Let x, y be coordinates on $\mathbb{R}^2$ and z_x, z_y the corresponding partial derivatives of $z \in M^0$. We define

$$\mathcal{D}z = z_x\, \xi + (\sigma_+ - I) z\, \tau \qquad \mathrm{D}z = \kappa\, \mathrm{e}^{2\phi}\, \sigma_- z\, \xi + (z_y + \phi_y\, \sigma_3 z)\, \tau \tag{2.2}$$

with a constant κ, the 2×2 unit matrix I and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Then $(M, \mathcal{D}, \mathrm{D})$ is a bicomplex for the Liouville equation

$$\phi_{xy} = \kappa\, \mathrm{e}^{2\phi}. \tag{2.4}$$

Let us now consider two such bicomplexes corresponding to two different choices κ_i, $i = 1, 2$, for κ and corresponding solutions ϕ_i. Then the DBT condition (1.11) reads $[\mathcal{D}, Q] = \lambda\, (\mathrm{D}_2\, Q - Q\, \mathrm{D}_1)$. With the ansatz (1.15) for Q, this becomes

$$\begin{aligned} &[\mathcal{D}, Q^{(0)}] = 0 \qquad [\mathcal{D}, Q^{(k)}] = \mathrm{D}_2\, Q^{(k-1)} - Q^{(k-1)}\, \mathrm{D}_1 \ (k = 1, \ldots, N) \\ &\mathrm{D}_2\, Q^{(N)} = Q^{(N)}\, \mathrm{D}_1. \end{aligned} \tag{2.5}$$

Taking for $Q^{(k)}$ general 2×2 matrices, the entries of which are functions of x and y, the first equation leads to

$$Q^{(0)} = a(y)\, I + b(y)\, \sigma_+ \tag{2.6}$$

with functions a, b which do not depend on x. The $k = 1$ DBT condition in particular requires $a_y = 0$, so that a must be a constant. Furthermore, it leads to

$$Q^{(1)} = f\, I - \tfrac{1}{2}[b_y + b\, (\phi_1 + \phi_2)_y]\, \sigma_3 + c(y)\, \sigma_+ + r\, \sigma_- \tag{2.7}$$

[16] In some examples, the ξ^r can be realized as differentials of coordinates on a manifold. In this way contact is made with ordinary zero-curvature formulations (linear in λ) of continuous integrable models. The generalization from the algebra of differential forms to an abstract Grassmann algebra is important, however, in order to treat relevant examples within the bicomplex framework.

with an arbitrary function $c(y)$ and functions f, r subject to

$$f_x = \frac{b}{2}\,(\kappa_2\,\mathrm{e}^{2\phi_2} - \kappa_1\,\mathrm{e}^{2\phi_1}) \tag{2.8}$$

$$r = a\,(\phi_2 - \phi_1)_y \qquad r_x = a\,[\kappa_2\,\mathrm{e}^{2\phi_2} - \kappa_1\,\mathrm{e}^{2\phi_1}]. \tag{2.9}$$

The next equation in (2.5) requires in particular[17] $r_y = r\,(\phi_1 + \phi_2)_y$ and thus

$$r = \alpha\,\mathrm{e}^{\phi_1+\phi_2} \tag{2.10}$$

where α does not depend on y. If $a \neq 0$, the two equations (2.9) now reproduce a well known BT for the Liouville equation:

$$(\phi_2 - \phi_1)_y = \alpha\,\mathrm{e}^{\phi_1+\phi_2} \qquad (\phi_1 + \phi_2)_x = \frac{1}{\alpha}\,(\kappa_2\,\mathrm{e}^{\phi_2-\phi_1} - \kappa_1\,\mathrm{e}^{\phi_1-\phi_2} - \alpha_x) \tag{2.11}$$

where a has been absorbed via a rescaling of α (cf [2], for example). This is precisely obtained as the primary DBT condition with

$$Q = I + \lambda\,\alpha\,\mathrm{e}^{\phi_1+\phi_2}\,\sigma_-. \tag{2.12}$$

The remaining freedom in the general solution for Q (where $N \geqslant 1$) can only restrict this BT. In particular, it cannot lead to different BTs. The case $a = 0$ and $b \neq 0$ does not lead to additional BTs either.

If $\kappa_1 = 1$ and $\kappa_2 = 0$, (2.11) is a hetero-BT connecting solutions of the Liouville equation $\phi_{xy} = \mathrm{e}^{2\phi}$ with solutions of $\phi_{xy} = 0$ (which is the wave equation in light cone coordinates).

Let us now explore the permutability conditions for $N = 1$. Using $\sigma_-^2 = 0$, (1.14) with (2.12) becomes

$$\phi_3 = \phi_0 + \ln\left(\frac{\alpha_{10}\,\mathrm{e}^{\phi_1} - \alpha_{20}\,\mathrm{e}^{\phi_2}}{\alpha_{32}\,\mathrm{e}^{\phi_2} - \alpha_{31}\,\mathrm{e}^{\phi_1}}\right). \tag{2.13}$$

This allows us to compute a solution ϕ_3 in a purely algebraic way from solutions ϕ_0, ϕ_1, ϕ_2, if the pairs (ϕ_0, ϕ_1) and (ϕ_0, ϕ_2) satisfy (2.11).

The fact that $N > 1$ does not lead to other BTs is a special feature of the Liouville example. Let us consider the case $\kappa_i = 1$, $i = 1, 2$, in more detail. In general, the higher DBTs should be expected to be compositions of primary DBTs (see also the KdV example in section 4.1). Indeed, in the following we show that the composition of two Liouville BTs is again of the form (2.11). Differentiating the Liouville equation (2.4) with respect to x, we find $\phi_{xxy} = 2\phi_x\phi_{xy}$ and an integration with respect to y leads to

$$\phi_{xx} = f(x) + {\phi_x}^2 \tag{2.14}$$

with integration 'constant' $f(x)$. Multiplying the first part of the BT (2.11) by $\mathrm{e}^{\phi_1-\phi_2}$, we obtain $-(\mathrm{e}^{\phi_1-\phi_2})_y = \alpha\,\mathrm{e}^{2\phi_1} = \alpha\,\phi_{1xy}$. Integration of the last equation yields

$$\phi_2 = \phi_1 - \ln(k - \alpha\,\phi_{1x}) \tag{2.15}$$

with integration 'constant' $k(x)$. The latter is determined by α via

$$(k/\alpha)_x + (1 - k^2)/\alpha^2 = f \tag{2.16}$$

which follows with the help of the second BT part in (2.11). Let us now consider two BTs with

$$\phi_2 = \phi_1 - \ln(k_{21} - \alpha_{21}\,\phi_{1x}) \qquad \phi_3 = \phi_2 - \ln(k_{32} - \alpha_{32}\,\phi_{2x}). \tag{2.17}$$

Eliminating ϕ_2 from the second equation with the help of the first, using (2.14) we find

$$\phi_3 = \phi_1 - \ln(k_{31} - \alpha_{31}\,\phi_{1x}) \tag{2.18}$$

[17] For $N = 1$ this is a consequence of the last equation in (2.5). For $N > 1$, it follows from the $k = 2$ equation.

where

$$\alpha_{31} = \alpha_{32}k_{21} + k_{32}\,(\alpha_{21} + \alpha_{21x}) \qquad k_{31} = k_{32}k_{21} + \alpha_{32}\,(k_{21x} - \alpha_{21} f_1) \tag{2.19}$$

solve (2.16) with $f_1 = \phi_{1xx} - {\phi_{1x}}^2$. Hence, composition of Liouville BTs preserves their form.

Remark. The infinitesimal version of the first of equations (2.11) is $\delta\phi_y = \delta\alpha\, e^{2\phi+\delta\phi} = \delta\alpha\, e^{2\phi}$ where δ denotes a variation. Using the Liouville equation with $\kappa = 1$, this can be integrated with respect to y, so that $\delta\phi = \phi_x\,\delta\alpha - \delta k$ where $\delta k(x)$ is an 'integration constant'. This is also obtained as the variation of (2.15) about $\alpha = 0$ and $k = 1$. Together with the variation of the Liouville equation, $\delta\phi_{xy} = 2\,e^{2\phi}\,\delta\phi$, we obtain $\delta k = -\frac{1}{2}\delta\alpha_x$ and thus $\delta\phi = \delta\alpha\,\phi_x + \frac{1}{2}\delta\alpha_x$. As a consequence, $[\delta_1, \delta_2]\,\phi = \delta\alpha_3\,\phi_x + \frac{1}{2}\,\delta\alpha_{3x}$ with $\delta\alpha_3 = \delta\alpha_1\,\delta\alpha_{2x} - \delta\alpha_2\,\delta\alpha_{1x}$.

Discrete Liouville equation. Let $M = M^0 \otimes \Lambda_2$ where M^0 is the set of maps $\mathbb{Z}^2 \to \mathbb{R}^2$. In terms of the shift operators $(S_x z)(x, y) = z(x+1, y)$ and $(S_y z)(x, y) = z(x, y+1)$, we define[18]

$$\begin{aligned} \mathcal{D}z &= (S_x z - z)\,\xi + (\sigma_+ S_y z - z)\,\tau \\ \mathrm{D}z &= \kappa\, e^{S_x\phi+\phi}\,\sigma_- S_x z\,\xi + \left(e^{(S_y\phi-\phi)\,\sigma_3}\, S_y z - z\right)\tau \end{aligned} \tag{2.20}$$

with a constant κ. Then $\mathcal{D}^2 = 0$ and $\mathrm{D}^2 = 0$ identically, and $\mathcal{D}\,\mathrm{D} + \mathrm{D}\,\mathcal{D} = 0$ turns out to be equivalent to

$$e^{-\phi(x+1,y)}e^{-\phi(x,y+1)} - e^{-\phi(x+1,y+1)}e^{-\phi(x,y)} = \kappa. \tag{2.21}$$

Introducing coordinates $u = (x+y)/\Delta$, $v = (x-y)/\Delta$ with $\Delta^2 = \kappa$ and $\phi(x, y) = -\varphi(u - \Delta, v)$, this reads

$$e^{\varphi(u,v-\Delta)}e^{\varphi(u,v+\Delta)} - e^{\varphi(u-\Delta,v)}e^{\varphi(u+\Delta,v)} = \Delta^2 \tag{2.22}$$

which is Hirota's discretization of the Liouville equation [24, 25]. Let us now explore the corresponding DBTs with $Q^{(0)} = I$. Then we have to solve the equation

$$[\mathcal{D}, Q^{(1)}] = \mathrm{D}_2 - \mathrm{D}_1 \tag{2.23}$$

which restricts $Q^{(1)}$ to the form $Q^{(1)} = r\,\sigma_-$ with a function r. Furthermore, we obtain the following set of equations:

$$r(x, y) = e^{\phi_1(x,y)-\phi_1(x,y+1)} - e^{\phi_2(x,y)-\phi_2(x,y+1)} \tag{2.24}$$

$$r(x, y+1) = e^{\phi_2(x,y+1)-\phi_2(x,y)} - e^{\phi_1(x,y+1)-\phi_1(x,y)} \tag{2.25}$$

$$r(x+1, y) - r(x, y) = \kappa_2\, e^{\phi_2(x+1,y)+\phi_2(x,y)} - \kappa_1\, e^{\phi_1(x+1,y)+\phi_1(x,y)}. \tag{2.26}$$

Using $\partial_{+y}\phi = \phi(x, y+1) - \phi(x, y)$, the first equation can be written as

$$r = e^{-\partial_{+y}\phi_1} - e^{-\partial_{+y}\phi_2} = (e^{\partial_{+y}\phi_2} - e^{\partial_{+y}\phi_1})\, e^{-\partial_{+y}(\phi_1+\phi_2)} \tag{2.27}$$

with the help of which we can convert (2.25) into

$$S_y r = r\, e^{\partial_{+y}(\phi_1+\phi_2)}. \tag{2.28}$$

This equation can be 'integrated' and yields

$$r = \alpha(x)\, e^{\phi_1+\phi_2} \tag{2.29}$$

with an arbitrary function $\alpha(x)$. Together with (2.24), it leads to the first part of the BT,

$$e^{-S_y\phi_1}\, e^{-\phi_2} - e^{-\phi_1}\, e^{-S_y\phi_2} = \alpha. \tag{2.30}$$

[18] Here and in the following we also use the shift operators acting on functions via $(S_x\phi)(x, y) = \phi(x+1, y)$.

The other BT part follows from (2.26):

$$(S_x\alpha)\,\mathrm{e}^{S_x\phi_1+S_x\phi_2} - \alpha\,\mathrm{e}^{\phi_1+\phi_2} = \kappa_2\,\mathrm{e}^{S_x\phi_2+\phi_2} - \kappa_1\,\mathrm{e}^{S_x\phi_1+\phi_1}. \tag{2.31}$$

For $N = 1$, no additional equations arise from the remaining DBT conditions. For $N > 1$, we could at most obtain restrictions on the above BT. In particular, no new BTs can show up, as in the case of the continuous Liouville equation. Taking (2.29) into account, the $N = 1$ expression for Q is the same as for the continuum model. Hence we obtain the same permutability condition.

3. Darboux–Bäcklund transformations of dressed bicomplexes

All the examples presented in the next section possess a somewhat more specialized form of the bicomplex equations than what is allowed by the general formalism of section 1. For this class one can make some general observations which help to reduce the number of calculations needed to elaborate the DBTs in concrete examples. The corresponding formalism is developed in this section. Clearly, this is of a more technical nature. In principle, given a bicomplex formulation of some equation, the formulae of section 1 are sufficient to work out the corresponding DBTs. In a given example, however, it may turn out to be very difficult to do it in a straightforward way.

It is often convenient [14, 15] to start with a trivial[19] bicomplex and to use what we call 'dressings' to construct nontrivial bicomplexes. Normally, such a 'deformation' of a trivial bicomplex results in too many independent equations, but there are two particular ways of introducing dressings (see cases (A) and (B) below) which keep some of the bicomplex equations identically satisfied.

Let (M, δ, d) be a trivial bicomplex and L the space of linear operator-valued forms[20] acting on M; i.e., for $z \in M$ and $T \in L$ we have $Tz \in M$. On operators we define

$$\tilde{\mathrm{d}}T = [\mathrm{d}, T] \qquad \tilde{\delta}T = [\delta, T] \tag{3.1}$$

where $[\,,\,]$ is the graded commutator[21]. Then $(L, \tilde{\delta}, \tilde{\mathrm{d}})$ is again a bicomplex and, moreover, a bi-differential calculus[22] [15]. A dressing of the bicomplex (M, δ, d) is a new bicomplex $(M, \mathcal{D}, \mathrm{D})$, where

$$\mathrm{D}z = \mathrm{d}z + A\,z \qquad \mathcal{D}z = \delta z + B\,z \tag{3.2}$$

with 'connection' one-forms $A, B \in L$. The conditions for $(M, \mathcal{D}, \mathrm{D})$ to be a bicomplex impose the following conditions on A and B:

$$F = \tilde{\mathrm{d}}A + A^2 = 0 \qquad \mathcal{F} = \tilde{\delta}B + B^2 = 0 \qquad \tilde{\delta}A + \tilde{\mathrm{d}}B + A\,B + BA = 0. \tag{3.3}$$

Introducing a real parameter λ and

$$\mathrm{d}_\lambda = \delta - \lambda\,\mathrm{d} \qquad A(\lambda) = B - \lambda\,A \tag{3.4}$$

the three conditions (3.3) can be compactly written as a λ-dependent zero-curvature condition,

$$F(\lambda) = \tilde{\mathrm{d}}_\lambda A(\lambda) + A(\lambda)^2 = 0 \tag{3.5}$$

for all λ.

[19] 'Trivial' in the sense that the corresponding bicomplex conditions are identically satisfied.
[20] The elements of M^s are called s-forms.
[21] δ and d are odd. For an even operator T, $[\mathrm{d}, T] = \mathrm{d}T - T\mathrm{d}$. For an odd T, $[\mathrm{d}, T] = \mathrm{d}T + T\mathrm{d}$.
[22] Besides $\tilde{\delta}^2 = \tilde{\mathrm{d}}^2 = \tilde{\mathrm{d}}\tilde{\delta} + \tilde{\delta}\tilde{\mathrm{d}} = 0$, $\tilde{\delta}$ and $\tilde{\mathrm{d}}$ obey the Leibniz rule, i.e. the graded product rule of differentiation.

Now we consider two dressings with A_i, B_i, $i = 1, 2$, and look for a DBT of the two resulting bicomplexes $\mathcal{B}_i$[23]. The DBT condition (1.11) then becomes

$$\tilde{\mathrm{d}}_\lambda Q + A_2(\lambda)\, Q - Q\, A_1(\lambda) = 0. \tag{3.6}$$

In terms of A and B, this reads

$$\tilde{\delta} Q + B_2\, Q - Q\, B_1 = \lambda\, \tilde{\mathrm{D}}_{21} Q \tag{3.7}$$

where

$$\tilde{\mathrm{D}}_{21} Q = \tilde{\mathrm{d}} Q + A_2\, Q - Q\, A_1. \tag{3.8}$$

In the following we confine our considerations to the case where $B_1 = 0 = B_2$[24]. Then the conditions (3.3) for the two bicomplexes reduce to

$$F_i = \tilde{\mathrm{d}} A_i + A_i^2 = 0 \qquad \tilde{\delta} A_i = 0 \tag{3.9}$$

and (3.7) becomes

$$\tilde{\delta} Q = \lambda\, \tilde{\mathrm{D}}_{21} Q. \tag{3.10}$$

This equation has to be solved in order to determine the DBTs of a dressed bicomplex. Using the ansatz (1.15), the DBT condition splits into the following set of equations:

$$\tilde{\delta} Q^{(0)} = 0 \qquad \tilde{\delta} Q^{(k)} = \tilde{\mathrm{D}}_{21} Q^{(k-1)} \quad (k = 1, \ldots, N) \qquad \tilde{\mathrm{D}}_{21} Q^{(N)} = 0. \tag{3.11}$$

Now we assume that Q is invertible with[25] $Q^{(0)} = I$ and consider in more detail the case of a primary DBT with

$$Q = I + \lambda\, R \tag{3.12}$$

where $R = Q^{(1)}$ does not depend on λ. Equation (3.11) then reduces to

$$\tilde{\delta} R = \tilde{\mathrm{D}}_{21} I = A_2 - A_1 \qquad \tilde{\mathrm{D}}_{21} R = 0. \tag{3.13}$$

In fact, in all the examples which we have explored so far, it turned out to be sufficient to consider such a primary invertible DBT in order to recover well known BTs. We can somewhat simplify the last set of equations as follows, using two obvious ways to reduce the set of bicomplex equations [14][26].

Case (A). Let $A_i = g_i^{-1} \tilde{\mathrm{d}} g_i$ where $g_i : M^0 \to M^0$ are invertible operators (e.g. matrices) not depending on λ. This solves the first of the bicomplex equations (3.9). The second of equations (3.13) is then equivalent to $\tilde{\mathrm{d}}(g_2 R g_1^{-1}) = 0$, which can be converted to $\tilde{\mathrm{d}} a = 0$ by setting $R = g_2^{-1}\, a\, g_1$ with[27] $a \in L^0$. The first of equations (3.13) now becomes

$$\tilde{\delta}(g_2^{-1} a\, g_1) = g_2^{-1} \tilde{\mathrm{d}} g_2 - g_1^{-1} \tilde{\mathrm{d}} g_1 = g_2^{-1} \tilde{\mathrm{d}}(g_2 g_1^{-1})\, g_1. \tag{3.14}$$

[23] Note that in this framework the linear spaces M_i of the two bicomplexes are taken to be the same. If Q is invertible, this is not a restriction as pointed out in section 1.

[24] Actually, in general, this can be achieved by separate gauge transformations and coordinate transformations of the two bicomplexes, since $\mathcal{F}_i = 0$. See section 5 for an example.

[25] This choice for $Q^{(0)}$ trivially solves the first of equations (3.11). In the case of auto-DBTs, arguments for this choice (under certain assumptions) have been given in section 1.

[26] The following is specific to bicomplexes and has no analogue in the case of a zero-curvature condition nonlinear in λ. Though such a condition can also be solved as in (A) by writing the λ-dependent gauge potential as a 'pure gauge', the corresponding g then depends on λ.

[27] For arbitrary N, this holds with R replaced by $Q^{(N)}$.

Case (B). Let $A_i = \tilde{\delta} w_i$ with $w_i \in L^0$ not depending on λ. This solves the second of the bicomplex equations (3.9). From the first equation of (3.13) we obtain

$$R = w_2 - w_1 + T \tag{3.15}$$

with $\tilde{\delta} T = 0$. Then the second equation in (3.13) becomes

$$\Phi_2 - \Phi_1 + \tilde{\mathrm{d}} T + \tilde{\delta}(w_2 T - T w_1 + \tfrac{1}{2}(w_2 - w_1)^2 + \tfrac{1}{2}[w_1, w_2]) = 0 \tag{3.16}$$

where

$$\Phi_i = \tilde{\mathrm{d}} w_i + \tfrac{1}{2}[\tilde{\delta} w_i, w_i] \tag{3.17}$$

are $\tilde{\delta}$-potentials of the curvatures F_i, i.e.

$$\tilde{\delta} \Phi_i = -F_i. \tag{3.18}$$

Let us assume that the first $\tilde{\delta}$-cohomology is trivial. The last equation together with $F_i = 0$ then implies

$$\Phi_i = \tilde{\delta} \Psi_i. \tag{3.19}$$

Furthermore, $\tilde{\delta}\tilde{\mathrm{d}} T = 0$ leads to $\tilde{\mathrm{d}} T = \tilde{\delta} b$ with $b \in L^0$. Then (3.16) can be integrated and leads to[28]

$$\Psi_2 - \Psi_1 + w_2 T - T w_1 + \tfrac{1}{2}(w_2 - w_1)^2 + \tfrac{1}{2}[w_1, w_2] + b = 0. \tag{3.20}$$

In concrete examples, it is often simpler to work out directly the second of equations (3.13), however.

For a DBT with Q of the form (1.15) where $Q^{(0)} = I$, the permutability condition (1.14) results in the following system of equations:

$$Q_{31}^{(1)} + Q_{10}^{(1)} - Q_{32}^{(1)} - Q_{20}^{(1)} = 0 \tag{3.21}$$

$$Q_{31}^{(k)} + Q_{10}^{(k)} - Q_{32}^{(k)} - Q_{20}^{(k)} = \sum_{m=1}^{k-1} \left(Q_{32}^{(m)} Q_{20}^{(k-m)} - Q_{31}^{(m)} Q_{10}^{(k-m)} \right) \qquad k = 2, \ldots, N \tag{3.22}$$

$$\sum_{\substack{m+n=k \\ 1 \leqslant m,n \leqslant N}} \left(Q_{31}^{(m)} Q_{10}^{(n)} - Q_{32}^{(m)} Q_{20}^{(n)} \right) = 0 \qquad k = N+1, \ldots, 2N. \tag{3.23}$$

For a primary DBT with $Q = I + \lambda R$, this reduces to

$$R_{31} + R_{10} = R_{32} + R_{20} \tag{3.24}$$

$$R_{31} R_{10} = R_{32} R_{20}. \tag{3.25}$$

In case (A), we have $R_{ij} = g_i^{-1} a_{ij} g_j$, so (3.25) becomes

$$a_{31} a_{10} = a_{32} a_{20}. \tag{3.26}$$

In case (B), we have $R_{ij} = \phi_i - \phi_j + T_{ij}$. Then (3.24) reduces to

$$T_{31} + T_{10} = T_{32} + T_{20}. \tag{3.27}$$

4. Bicomplexes and auto-Darboux–Bäcklund transformations for various integrable models

In the following, we elaborate auto-DBTs for various integrable models. All the examples of nontrivial bicomplexes in this section are of the form (M, δ, D) where (M, δ, d) is a trivial bicomplex and D has the decomposed 'dressed' form $\mathrm{D} = \mathrm{d} + A$ as considered in the previous section. Assuming Q to be invertible, in section 1 we motivated a restriction of the auto-DBT condition to the form (1.17) with $Q^{(0)} = I$. This is the basis for the following calculations. As in section 2, Λ_2 denotes the exterior algebra of a two-dimensional real vector space.

[28] An 'integration constant' c with $\tilde{\delta} c = 0$ can be absorbed by a redefinition of b.

4.1. KdV and related equations

4.1.1. KdV equation, primary DBT. Let $M = C^\infty(\mathbb{R}^2, \mathbb{R}) \otimes \Lambda_2$ with

$$\delta z = -3z_{xx}\,\tau + z_x\,\xi \qquad \mathrm{d}z = (z_t + 4z_{xxx})\,\tau - z_{xx}\,\xi \tag{4.1}$$

for $z \in M^0$. Dressed with the gauge potential one-form

$$A = \tilde{\delta} w = -3\,(w_{xx} + 2w_x\,\partial_x)\,\tau + w_x\,\xi \tag{4.2}$$

we obtain a bicomplex for the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{4.3}$$

where $u = w_x$ (see also [16]). Here we choose $w \in L^0$ as a function (which acts by multiplication). Looking for a primary DBT, we have (3.15) with $T_x = 0$. Furthermore,

$$\begin{aligned}\tilde{\mathrm{D}}_{21} R = \{&R_t + 4R_{xxx} + 12R_{xx}\,\partial_x + 12R_x\,\partial_x^2 - 3\,[(w_{2xx} + 2w_{2x}\,\partial_x)\,R \\ &-R\,(w_{1xx} + 2w_{1x}\,\partial_x)]\}\,\tau - (R_{xx} + 2R_x\,\partial_x - w_{2x}\,R + R\,w_{1x})\,\xi\end{aligned} \tag{4.4}$$

has to vanish. Using (3.15), the ξ-part leads to

$$(w_2 - w_1)_{xx} + 2(w_2 - w_1)_x\,\partial_x - w_{2x}\,(w_2 - w_1 + T) + (w_2 - w_1 + T)\,w_{1x} = 0. \tag{4.5}$$

In particular, this implies $T = 2\partial_x + \beta$ with a function $\beta(t)$, which, however, can be absorbed via a redefinition of w_1 (which leaves the KdV equation invariant). Hence

$$R = w_2 - w_1 + 2\partial_x \tag{4.6}$$

and (4.5) leads to the BT part

$$(w_1 + w_2)_x = 2\alpha + \tfrac{1}{2}\,(w_2 - w_1)^2 \tag{4.7}$$

with an integration 'constant' α which is an arbitrary function of t. The vanishing of the τ-part of (4.4) together with (4.7) yields[29]

$$(w_2 - w_1)_t + (w_2 - w_1)_{xxx} - 3(w_2 - w_1)_x\,(w_1 + w_2)_x = 0 \tag{4.8}$$

which is the second BT part for the KdV equation (see [5], p 113, for example, and also [2, 3, 26, 27]). Introducing

$$r = w_2 - w_1 \tag{4.9}$$

the BT can also be written as

$$w_{1x} = \alpha - \tfrac{1}{2}\,r_x + \tfrac{1}{4}\,r^2 \qquad r_t + [r_{xx} - \tfrac{1}{2}r^3 - 6\alpha r]_x = 0. \tag{4.10}$$

The last equation has the form of a conservation law and can be integrated once if we set

$$r = -2\,(\ln\chi)_x. \tag{4.11}$$

Then, in terms of χ, the BT reads

$$w_{1x} = \alpha + \frac{\chi_{xx}}{\chi} \qquad \frac{\chi_t}{\chi_x} = 6\alpha - \frac{\chi_{xxx}}{\chi_x} + 3\frac{\chi_{xx}}{\chi} + \gamma\,\frac{\chi}{\chi_x} \tag{4.12}$$

with a 'constant of integration' $\gamma(t)$, assuming $\chi_x \neq 0$.

The following will be needed below for our discussion of the secondary DBT. We consider two BTs,

$$(w_1 + w_2)_x = 2\alpha_1 + \tfrac{1}{2}(w_1 - w_2)^2 \qquad (w_2 + w_3)_x = 2\alpha_2 + \tfrac{1}{2}(w_2 - w_3)^2. \tag{4.13}$$

[29] This is also obtained by integration of the difference of the two KdV equations for $u_1 = w_{1x}$ and $u_2 = w_{2x}$ with vanishing integration 'constant'.

Subtracting the second equation from the first, we obtain

$$(w_1 - w_3)_x = 2(\alpha_1 - \alpha_2) + \tfrac{1}{2}(w_1 - w_3)(w_1 + w_3 - 2w_2). \tag{4.14}$$

Solving for w_2 yields

$$w_2 = \frac{1}{2}(w_1 + w_3) - \frac{s_x}{s} + \frac{2(\alpha_1 - \alpha_2)}{s} \qquad s = w_1 - w_3. \tag{4.15}$$

Inserting this expression into the sum of the two equations (4.13), we obtain

$$(w_1 + w_3)_x = \alpha_1 + \alpha_2 + \frac{s_{xx}}{s} - \frac{1}{2}\left(\frac{s_x}{s}\right)^2 + \frac{1}{8}s^2 + \frac{2(\alpha_1 - \alpha_2)^2}{s^2}. \tag{4.16}$$

The complementary parts of the above two BTs are

$$(w_1 - w_2)_t + (w_1 - w_2)_{xxx} - 3\,(w_{1x}^2 - w_{2x}^2) = 0 \tag{4.17}$$

$$(w_2 - w_3)_t + (w_2 - w_3)_{xxx} - 3\,(w_{2x}^2 - w_{3x}^2) = 0. \tag{4.18}$$

Adding these two equations leads to

$$s_t + s_{xxx} - 3s_x\,(w_1 + w_3)_x = 0. \tag{4.19}$$

Using (4.16) to eliminate $(w_1 + w_3)_x$, we obtain

$$s_t + \left[s_{xx} - 3(\alpha_1 + \alpha_2)\,s - \frac{1}{8}s^3 + 6(\alpha_1 + \alpha_2)^2\frac{1}{s} - \frac{3s_x^2}{2s}\right]_x = 0 \tag{4.20}$$

which, setting $s = -2(\ln\chi)_x$ and integrating once with integration 'constant' $2\epsilon(t)$, becomes

$$\frac{\chi_t}{\chi_x} = 3(\alpha_1 + \alpha_2) - \mathcal{S}_x\chi - \frac{3(\alpha_1 - \alpha_2)^2}{2}\left(\frac{\chi}{\chi_x}\right)^2 + \epsilon\frac{\chi}{\chi_x}. \tag{4.21}$$

Here, $\mathcal{S}_x$ denotes the Schwarzian derivative

$$\mathcal{S}_x\chi = \frac{\chi_{xxx}}{\chi_x} - \frac{3}{2}\left(\frac{\chi_{xx}}{\chi_x}\right)^2. \tag{4.22}$$

Correspondingly, (4.16) can be rewritten in the form

$$w_{3x} = \frac{\alpha_1 + \alpha_2}{2} + \frac{\chi_{xxx}}{2\chi_x} - \left(\frac{\chi_{xx}}{2\chi_x}\right)^2 + \frac{(\alpha_1 - \alpha_2)^2}{4}\left(\frac{\chi}{\chi_x}\right)^2. \tag{4.23}$$

From these expressions one recovers for $\alpha_1 = \alpha_2$ and $\epsilon = 0$ a BT found by Galas [28]. The latter is therefore just the composition of two 'elementary' BTs.

4.1.2. KdV equation, permutability. With (4.6), the permutability condition (3.24) is identically satisfied and (3.25) leads to

$$w_3 = -w_0 + w_1 + w_2 - \frac{2\,(w_1 - w_2)_x}{w_1 - w_2}. \tag{4.24}$$

With the help of (4.7), for the pairs w_0, w_1 with α_1 and w_0, w_2 with α_2, the last equation can be written as

$$w_3 = w_0 - 4\,\frac{\alpha_1 - \alpha_2}{w_1 - w_2}. \tag{4.25}$$

4.1.3. KdV equation, secondary DBT. We consider again the above bicomplex associated with the KdV equation, but now we turn to the secondary DBT. Again, we have $Q^{(1)} = w_2 - w_1 + T$ with $T_x = 0$. For $N = 2$, (3.11) then requires

$$\tilde{\delta} Q^{(2)} = \tilde{\mathrm{d}}(w_2 - w_1 + T) + (\tilde{\delta} w_2)\, w_2 + w_1\, \tilde{\delta} w_1 + \tilde{\delta}(-w_1 w_2 + w_2 T - T w_1) \tag{4.26}$$

$$\tilde{\mathrm{d}} Q^{(2)} = Q^{(2)}\, \tilde{\delta} w_1 - (\tilde{\delta} w_2)\, Q^{(2)}. \tag{4.27}$$

The ξ-part of (4.26) can be integrated and leads to

$$Q^{(2)} = (w_1 - w_2)_x + \tfrac{1}{2}(w_1 - w_2)^2 + 2(w_1 - w_2)\,\partial_x + w_2 T - T w_1 + \rho \tag{4.28}$$

where $\rho_x = 0$. Inserted in the ξ-part of (4.27), which is

$$Q^{(2)}_{xx} + Q^{(2)}\, w_{1x} - w_{2x}\, Q^{(2)} + 2\, Q^{(2)}_x\, \partial_x = 0 \tag{4.29}$$

this enforces $T = 4\partial_x + \beta(t)$. The function $\beta(t)$ can be absorbed by a redefinition of w_1. Furthermore, we obtain

$$Q^{(2)} = -r_x + \tfrac{1}{2} r^2 - 4\, w_{1x} + 2r\, \partial_x + 4\, \partial_x^2 + 4\alpha \tag{4.30}$$

with $r = w_2 - w_1$ and an arbitrary function $\alpha(t)$, and finally

$$(w_1 + w_2)_x = 2\alpha + \frac{r_{xx}}{r} - \frac{1}{2}\left(\frac{r_x}{r}\right)^2 + \frac{1}{8} r^2 - \frac{8\gamma}{r^2} \tag{4.31}$$

with an integration 'constant' $\gamma(t)$. The τ-part of (4.26) is now evaluated to

$$r_t + r_{xxx} - 3r_x\,(r_x + 2w_{1x}) = 0. \tag{4.32}$$

Elimination of w_{1x} with the help of (4.31) leads to

$$r_t + \left[r_{xx} - \frac{3}{2}\frac{r_x^2}{r} - 6\alpha\, r - \frac{1}{8} r^3 - 24\frac{\gamma}{r} \right]_x = 0. \tag{4.33}$$

Setting $r = -2(\ln \chi)_x$, this equation can be integrated and rewritten as

$$\frac{\chi_t}{\chi_x} = 6\alpha - \mathcal{S}_x\, \chi + 6\gamma \left(\frac{\chi}{\chi_x}\right)^2 + \epsilon \frac{\chi}{\chi_x} \tag{4.34}$$

with an arbitrary function $\epsilon(t)$ and the Schwarzian derivative defined in (4.22). In terms of χ, (4.31) takes the form

$$w_{1x} = \alpha + \frac{\chi_{xxx}}{2\chi_x} - \left(\frac{\chi_{xx}}{2\chi_x}\right)^2 - \gamma \left(\frac{\chi}{\chi_x}\right)^2. \tag{4.35}$$

Comparison with our previous results shows that this secondary DBT for the KdV bicomplex is just the composition of two primary DBTs. If $\gamma < 0$, this is evident. If $\gamma > 0$ we set $\alpha_1 = \alpha + \mathrm{i}\sqrt{\gamma}$ and $\alpha_2 = \alpha - \mathrm{i}\sqrt{\gamma}$. Although in this case the elementary BTs (with α_1 and α_2, respectively) do not produce *real* solutions from real solutions, in general, their composition does.

4.1.4. Modified KdV equation. Supplied with the product $(a, b)(a', b') = (aa', bb')$, the vector space $\mathbb{R}^2$ becomes a commutative ring with unit $(1, 1)$, which we denote as ${}^2\mathbb{R}$ [29]. It is a realization of the abstract commutative ring generated by a unit 1 and another element e satisfying $e^2 = 1$. Here we have $e = (1, -1)$. The relevance of ${}^2\mathbb{R}$ for the mKdV equation stems from the following observation (see also [30]). Let u be a field with values in ${}^2\mathbb{R}$ which satisfies the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{4.36}$$

and let

$$u = v^2 + v_x\, e \tag{4.37}$$

with a real-valued field v. Now

$$u_t + u_{xxx} - 6uu_x = (2v + e\,\partial_x)(v_t + v_{xxx} - 6v^2 v_x) \tag{4.38}$$

shows that the field v satisfies the mKdV equation

$$v_t + v_{xxx} - 6v^2 v_x = 0. \tag{4.39}$$

In fact, (4.37) is the famous Miura transformation and its 'conjugate' since with $u = (u^+, u^-) = (v^2, v^2) + (v_x, -v_x)$ we obtain

$$u^+ = v_x + v^2 \qquad u^- = -v_x + v^2. \tag{4.40}$$

Consequently, the two KdV equations for $u^\pm$ are equivalent to the above mKdV equation. Hence, in order to find an auto-BT for the mKdV equation, we simply have to extend our KdV treatment to fields with values in ${}^2\mathbb{R}$, though we have to take care of the fact that ${}^2\mathbb{R}$ is not a division ring (since $(1, 0)(0, 1) = (1+e)(1-e) = 0$). Introducing $w = \hat{v} + v\, e$ with $\hat{v}_x = v^2$, we have $w_x = u$ and we can directly generalize the KdV auto-BT:

$$(w_1 + w_2)_x - 2\alpha - \tfrac{1}{2}(w_1 - w_2)^2 = 0 \tag{4.41}$$

$$(w_1 - w_2)_t + (w_1 - w_2)_{xxx} - 3(w_1 - w_2)_x\,(w_1 + w_2)_x = 0 \tag{4.42}$$

(where $\beta(t)$ has been absorbed in w_1). Note that the integration 'constant' $\alpha(t)$ is now an element of ${}^2\mathbb{R}$. We decompose (4.41) with $\alpha = -k^2 + be$ to obtain the two equations

$$(\hat{v}_1 + \hat{v}_2)_x = -2k^2 + \tfrac{1}{2}[(\hat{v}_1 - \hat{v}_2)^2 + (v_1 - v_2)^2] \tag{4.43}$$

$$(v_1 + v_2)_x = 2b + (\hat{v}_1 - \hat{v}_2)(v_1 - v_2). \tag{4.44}$$

Using $\hat{v}_x = v^2$ in (4.43), we obtain

$$(v_1 + v_2)^2 = -4k^2 + (\hat{v}_1 - \hat{v}_2)^2. \tag{4.45}$$

Applying ∂_x to this equation and comparing the result with (4.44), we find $b = 0$. Solving the last equation for $\hat{v}_1 - \hat{v}_2$ and inserting this expression in (4.44) leads to the first part of the auto-BT[30],

$$(v_1 + v_2)_x = \pm(v_1 - v_2)\sqrt{4k^2 + (v_1 + v_2)^2}. \tag{4.46}$$

Decomposition of (4.42) leads to

$$(\hat{v}_1 - \hat{v}_2)_t + (\hat{v}_1 - \hat{v}_2)_{xxx} - 3\,(\hat{v}_1 - \hat{v}_2)_x\,(\hat{v}_1 + \hat{v}_2)_x - 3\,(v_1 - v_2)_x\,(v_1 + v_2)_x = 0 \tag{4.47}$$

$$(v_1 - v_2)_t + (v_1 - v_2)_{xxx} - 3(\hat{v}_1 - \hat{v}_2)_x\,(v_1 + v_2)_x - 3\,(v_1 - v_2)_x\,(\hat{v}_1 + \hat{v}_2)_x = 0. \tag{4.48}$$

Using $\hat{v}_x = v^2$ in the second equation produces the second part of the mKdV auto-BT,

$$(v_1 - v_2)_t + [(v_1 - v_2)_{xx} - 2\,v_1^3 + 2\,v_2^3]_x = 0 \tag{4.49}$$

which is the difference of the two mKdV equations for v_1 and v_2.

[30] With $v = q_x$, this reads $(q_1 + q_2)_x = 2k\,\sinh(q_1 - q_2)$ where an integration constant has been absorbed by a redefinition of the q_i.

4.1.5. mKdV equation, permutability. In the framework of the previous subsection, the KdV permutability condition takes the form

$$(w_2 - w_1)(w_1 + w_2 - w_0 - w_3) = 2\,(w_2 - w_1)_x \tag{4.50}$$

(cf (4.24)) and, by use of (4.41),

$$(w_3 - w_0)(w_2 - w_1) = 4\,(k_2^2 - k_1^2) \tag{4.51}$$

from which we obtain, by decomposition,

$$(v_3 - v_0)(\hat{v}_2 - \hat{v}_1) + (v_2 - v_1)(\hat{v}_3 - \hat{v}_0) = 0 \tag{4.52}$$

$$(\hat{v}_3 - \hat{v}_0)(\hat{v}_2 - \hat{v}_1) + (v_3 - v_0)(v_2 - v_1) = 4\,(k_2^2 - k_1^2) \tag{4.53}$$

and thus the following superposition formula for mKdV solutions:

$$v_3 = v_0 + \frac{4(k_2^2 - k_1^2)(v_2 - v_1)}{(v_2 - v_1)^2 - (\hat{v}_2 - \hat{v}_1)^2}. \tag{4.54}$$

4.1.6. ncKdV equation. We choose the bicomplex maps and the dressing as for the KdV equation, so that (4.1) and (4.2) hold, but now we take u to be a map from $\mathbb{R}^2$ into some noncommutative associative algebra with product $*$ for which ∂_t and ∂_x are derivations. Then we have a bicomplex iff $\tilde{\mathrm{d}}A + A * A = 0$ which is equivalent to the *noncommutative KdV equation* (ncKdV)

$$u_t + u_{xxx} - 3(u * u_x + u_x * u) = 0 \tag{4.55}$$

where $u = w_x$ [17]. The corresponding potential ncKdV equation is

$$w_t + w_{xxx} - 3w_x * w_x = 0. \tag{4.56}$$

The conditions for a primary DBT with Q of the form (3.12) are

$$\tilde{\delta} R = A_2 - A_1 \qquad \tilde{\mathrm{d}} R + A_2 * R - R * A_1 = 0. \tag{4.57}$$

As in the commutative case, the first equation is solved by $R = w_2 - w_1 + T$ with $T_x = 0$. The second equation implies $T = 2\partial_x + \beta(t)$. Again, the function $\beta(t)$ expresses the freedom in the choice of the potential for u and can be set to zero. Furthermore, we obtain

$$(w_1 + w_2)_{xx} - w_1 * w_{1x} - w_{2x} * w_2 + (w_2 * w_1)_x = 0 \tag{4.58}$$

$$(w_2 - w_1)_t + (w_2 - w_1)_{xxx} - 3(w_{2x} * w_{2x} - w_{1x} * w_{1x}) = 0. \tag{4.59}$$

In the commutative case, (4.58) can be integrated (which introduces a parameter in the BT). This is not so in the noncommutative case. We still have a BT which is no longer symmetric in w_1 and w_2, however.

4.1.7. ncKdV equation, permutability. The permutability conditions are reduced to $R_{31} * R_{10} = R_{32} * R_{20}$. This yields

$$w_3 * (w_1 - w_2) + (w_1 - w_2) * w_0 = w_1 * w_1 - w_2 * w_2 - 2(w_1 - w_2)_x \tag{4.60}$$

where each of the pairs (w_0, w_1), (w_0, w_2), (w_1, w_3), (w_2, w_3) has to satisfy the BT equations (4.58) and (4.59). Let us now specify the $*$-product as the Moyal product

$$f * h = \boldsymbol{m} \circ \mathrm{e}^{\vartheta P/2}(f \otimes h) \tag{4.61}$$

for smooth functions f, h, where $\boldsymbol{m}(f \otimes h) = fh$, $P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t$ and ϑ is a deformation parameter. As an example, let $w_3 = 0$, so that

$$(w_1 - w_2) * w_0 = w_1 * w_1 - w_2 * w_2 - 2(w_1 - w_2)_x. \tag{4.62}$$

For w_1 and w_2 we choose the one-soliton solutions

$$w_1 = -2\,\tanh(x - 4t) \qquad w_2 = -4\,\coth(2x - 32t) \tag{4.63}$$

(see also [5], p 116). (w_1, w_3) and (w_2, w_3) indeed satisfy (4.58) and (4.59). Then the $*$-products on the right side of (4.62) reduce to ordinary products and we obtain

$$f(x,t) * w_0 = g(x,t) \tag{4.64}$$

with

$$g(x,t) = 4\,\tanh^2(x - 4t) - 16\,\coth^2(2x - 32t) - 2\,f_x \tag{4.65}$$

$$f(x,t) = -2\,\tanh(x - 4t) + 4\,\coth(2x - 32t). \tag{4.66}$$

This implies

$$0 = \frac{\partial}{\partial\vartheta}(f * w_0) = f * \frac{\partial w_0}{\partial\vartheta} + \frac{1}{2}(f_t * w_{0x} - f_x * w_{0t}) \tag{4.67}$$

and

$$f * \frac{\partial^2 w_0}{\partial\vartheta^2} = -\left[f_t * \frac{\partial w_{0x}}{\partial\vartheta} - f_x * \frac{\partial w_{0t}}{\partial\vartheta}\right] - \frac{1}{4}(f_{tt} * w_{0xx} - 2f_{tx} * w_{0xt} + f_{xx} * w_{0tt}). \tag{4.68}$$

For vanishing deformation parameter ϑ, the solution of the permutability conditions is the two-soliton solution

$$W_0 = -6/[2\,\coth(2\,x - 32\,t) - \tanh(x - 4\,t)] \tag{4.69}$$

with corresponding KdV solution

$$u_0 = W_{0x} = -12\,\frac{3 + \cosh(4\,x - 64\,t) + 4\,\cosh(2\,x - 8\,t)}{(\cosh(3\,x - 36\,t) + 3\,\cosh(x - 28\,t))^2} \tag{4.70}$$

(cf [5], p 116). The noncommutative solution is then

$$w_0 = W_0 + \vartheta\,W_1 + \tfrac{1}{2}\vartheta^2\,W_2 + \cdots \tag{4.71}$$

where

$$W_1 := \left(\frac{\partial w_0}{\partial\vartheta}\right)_{\vartheta=0} = -\frac{1}{2f}(f_t\,W_{0x} - f_x\,W_{0t}) \tag{4.72}$$

$$W_2 := \left(\frac{\partial^2 w_0}{\partial\vartheta^2}\right)_{\vartheta=0} = -\frac{1}{f}(f_t\,W_{1x} - f_x\,W_{1t}) - \frac{1}{4f}(f_{tt}\,W_{0xx} - 2f_{tx}\,W_{0xt} + f_{xx}\,W_{0tt}) \tag{4.73}$$

and so forth. In the case under consideration, we obtain $W_1 = 0$ and

$$W_{2x} = 331\,776\,\frac{(\cosh(3\,x - 36\,t) - 3\,\cosh(x - 28\,t))\,(\sinh(3\,x - 36\,t) + \sinh(x - 28\,t))^2}{(\cosh(3\,x - 36\,t) + 3\,\cosh(x - 28\,t))^5} \tag{4.74}$$

which is precisely the expression for the second-order ncKdV correction u_2 to the classical two-soliton solution (4.70), obtained in [17] in a different way.

4.2. Further examples

4.2.1. Sine–Gordon equation. Let $M = C^\infty(\mathbb{R}^2, \mathbb{C}) \otimes \Lambda_2$ and

$$\delta z = z_x\,\xi + \tfrac{1}{2}(\bar z - z)\,\tau \qquad \mathrm{d}z = \tfrac{1}{2}(\bar z - z)\,\xi + z_y\,\tau \tag{4.75}$$

for $z \in M^0$, where a bar indicates complex conjugation. Dressing d with $A = g^{-1}\tilde{\mathrm{d}}g$, where $g = \mathrm{e}^{-\mathrm{i}\phi/2}$ with a real function ϕ, we obtain the map $\mathrm{D}z = \mathrm{e}^{\mathrm{i}\phi/2}\,\mathrm{d}(\mathrm{e}^{-\mathrm{i}\phi/2}z)$. Then (M, δ, D) is a bicomplex associated with the sine–Gordon equation $\phi_{xy} = \sin\phi$ [14]. Following scheme (A) of section 3, in order to calculate the primary DBT, we have to evaluate (3.14) with $g_i = \mathrm{e}^{-\mathrm{i}\phi_i/2}$ and some operator a. The latter has to satisfy $\tilde{\mathrm{d}}a = 0$, which means $\overline{az} = a\bar z$ and $a_y = 0$. First we calculate the right-hand side of (3.14):

$$g_2^{-1}\tilde{\mathrm{d}}(g_2 g_1^{-1})\,g_1\,z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\phi_2} - \mathrm{e}^{\mathrm{i}\phi_1})\,\bar z\,\xi - \frac{\mathrm{i}}{2}(\phi_2 - \phi_1)_y\,z\,\tau. \tag{4.76}$$

In order for this to be consistent (for all z) with the left-hand side of (3.14), the operator a must have the form $az = \alpha\,\bar z$ with a real function $\alpha(x)$. Then

$$\tilde\delta(g_2^{-1}a\,g_1)\,z = \left[\frac{\mathrm{i}\alpha}{2}\,(\phi_1 + \phi_2)_x + \alpha_x\right]\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)/2}\,\bar z\,\xi - \mathrm{i}\alpha\,\sin\left(\frac{\phi_1+\phi_2}{2}\right) z\,\tau \tag{4.77}$$

and (3.14) results in the following two equations:

$$(\phi_1+\phi_2)_x = \frac{2}{\alpha}\,\sin\frac{\phi_2-\phi_1}{2} + 2\mathrm{i}\,\frac{\alpha_x}{\alpha} \qquad (\phi_2-\phi_1)_y = 2\alpha\,\sin\frac{\phi_1+\phi_2}{2}. \tag{4.78}$$

Since we consider only *real* sine–Gordon solutions, we have to set $\alpha_x = 0$. Hence α has to be a real constant. Now we recover a famous auto-BT of the sine–Gordon equation [1–3, 31].

Since

$$R_{ij}\,z = \alpha_{ij}\,\mathrm{e}^{\mathrm{i}(\phi_i+\phi_j)/2}\,\bar z \tag{4.79}$$

the permutability condition (3.26) is satisfied with $\alpha_{10} = \alpha_{32} = \alpha_1$ and $\alpha_{20} = \alpha_{31} = \alpha_2$. The remaining permutability condition (3.24) requires

$$\alpha_1\,\mathrm{e}^{\mathrm{i}(\phi_0+\phi_1)/2} + \alpha_2\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_3)/2} - \alpha_2\,\mathrm{e}^{\mathrm{i}(\phi_0+\phi_2)/2} - \alpha_1\,\mathrm{e}^{\mathrm{i}(\phi_2+\phi_3)/2} = 0 \tag{4.80}$$

which is equivalent to Bianchi's 'permutability theorem' for the sine–Gordon equation:

$$\phi_3 = \phi_0 + 4\arctan\left[\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}\tan\left(\frac{\phi_1-\phi_2}{4}\right)\right]. \tag{4.81}$$

This determines a solution ϕ_3, if ϕ_1 and ϕ_2 are obtained from ϕ_0 via the BT, i.e. the pairs (ϕ_0, ϕ_1) and (ϕ_0, ϕ_2) have to satisfy (4.78)[31].

4.2.2. An equation related to the sine–Gordon equation. Let us again consider the trivial bicomplex (4.75) which we used in the context of the sine–Gordon equation. Now, however, we choose a different dressing:

$$\mathrm{D}z = \mathrm{d}z + [\delta, U]\,z = \tfrac{1}{2}[(1+2u_x)\,\bar z - z]\,\xi + [z_y + \tfrac{1}{2}(\bar u - u)\,z]\,\tau \tag{4.82}$$

where $Uz = u\,\bar z$ with a field $u(x, y)$. Then we have $\delta^2 = 0 = \delta\mathrm{D} + \mathrm{D}\delta$ identically, while $\mathrm{D}^2 = 0$ is equivalent to

$$u_{xy} = \tfrac{1}{2}(u - \bar u)(1 + 2u_x). \tag{4.83}$$

[31] The reader should be aware of a problem of notation in this section. A BT such as (4.78) is written in terms of solutions ϕ_1 and ϕ_2, but these are *not* the solutions ϕ_1 and ϕ_2 which appear in the permutability relations. See also section 1.

Since $\mathrm{D} = \mathrm{d} + \tilde{\delta} U$, following scheme (B) of section 3 we have $R = U_2 - U_1 + T$ with $\tilde{\delta} T = 0$ for a primary DBT. The latter condition requires $T_x = 0$ and $\overline{Tz} = T\bar{z}$. This is satisfied with $Tz = \alpha\,\bar{z}$ where $\alpha(y)$ is a real function. However, since the transformation $u_1 \mapsto u_1 + \alpha$ leaves (4.83) invariant, we may set $\alpha = 0$ and obtain

$$Rz = (u_2 - u_1)\,\bar{z}. \tag{4.84}$$

The primary DBT conditions now take the form

$$(\bar{u}_2 - \bar{u}_1)(1 + 2\,u_{2x}) - (u_2 - u_1)(1 + 2\,\bar{u}_{1x}) = 0 \tag{4.85}$$

$$(u_2 - u_1)_y + \tfrac{1}{2}\,(\bar{u}_1 + \bar{u}_2 - u_1 - u_2)(u_2 - u_1) = 0. \tag{4.86}$$

Adding the first equation to, respectively subtracting it from its complex conjugate, we deduce

$$|u_2 - u_1|_x = 0 \qquad \frac{\bar{u}_2 - \bar{u}_1}{u_2 - u_1} = \frac{1 + (\bar{u}_1 + \bar{u}_2)_x}{1 + (u_1 + u_2)_x}. \tag{4.87}$$

Furthermore, one obtains $|1 + 2\,u_{1x}| = |1 + 2\,u_{2x}|$.

The first permutability condition (3.24) is identically satisfied and the second permutability condition (3.25) leads to

$$u_3 = \frac{u_2\,(\bar{u}_2 - \bar{u}_0) - u_1\,(\bar{u}_1 - \bar{u}_0)}{\bar{u}_2 - \bar{u}_1}. \tag{4.88}$$

Comparing the operator D with the corresponding operator in the sine–Gordon case, we find the transformation

$$u_x = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\phi} - 1) \qquad \phi_y = \mathrm{i}\,(\bar{u} - u). \tag{4.89}$$

Eliminating ϕ from the above equations, we obtain (4.83). If we eliminate u, then we obtain the sine–Gordon equation.

4.2.3. Discrete sine–Gordon equation. Let M^0 be the space of complex functions on an infinite plane square lattice and

$$(\delta z)_S = (z_E - z_S)\,\xi + \kappa\,(\bar{z}_W - z_S)\,\tau \qquad (\mathrm{d}z)_S = \kappa\,(\bar{z}_E - z_S)\,\xi + (z_W - z_S)\,\tau \tag{4.90}$$

where κ is a real parameter. We use the notation $z_S = z(x-1, y-1)$, $z_E = z(x-1, y+1)$, $z_W = z(x+1, y-1)$, $z_N = z(x+1, y+1)$ (see also [14]). Now d is dressed with

$$(Az)_S = [(\mathrm{e}^{\mathrm{i}\phi/2}\,\tilde{\mathrm{d}}\mathrm{e}^{-\mathrm{i}\phi/2})\,z]_S = \kappa\,\mathrm{e}^{\mathrm{i}(\phi_E+\phi_S)/2}\,\bar{z}_E\,\xi + \mathrm{e}^{-\mathrm{i}(\phi_W-\phi_S)/2}\,z_W\,\tau. \tag{4.91}$$

The bicomplex condition $\tilde{\delta} A = 0$ then reads

$$\mathrm{e}^{\mathrm{i}(\phi_E-\phi_N)/2} - \mathrm{e}^{\mathrm{i}(\phi_S-\phi_W)/2} = \kappa^2\big(\mathrm{e}^{\mathrm{i}(\phi_W+\phi_N)/2} - \mathrm{e}^{\mathrm{i}(\phi_E+\phi_S)/2}\big) \tag{4.92}$$

which, multiplied by $\mathrm{e}^{\mathrm{i}(\phi_N-\phi_E+\phi_W-\phi_S)/4}$, produces the discrete sine–Gordon equation [32]

$$\sin[(\phi_N - \phi_E - \phi_W + \phi_S)/4] = \kappa^2\,\sin[(\phi_N + \phi_E + \phi_W + \phi_S)/4]. \tag{4.93}$$

Following scheme (A) of section 3 with $az = \alpha\,\bar{z}$ where α is a real constant, we find

$$(Rz)_S = (g_2^{-1}\,a\,g_1\,z)_S = \alpha\,\mathrm{e}^{\mathrm{i}(\phi_{1,S}+\phi_{2,S})/2}\,\bar{z}_S. \tag{4.94}$$

Now (3.14) generates the BT

$$\sin\frac{(\phi_1+\phi_2)_E - (\phi_1+\phi_2)_S}{4} = \frac{\kappa}{\alpha}\,\sin\frac{(\phi_2-\phi_1)_E + (\phi_2-\phi_1)_S}{4} \tag{4.95}$$

$$\sin\frac{(\phi_2-\phi_1)_W - (\phi_2-\phi_1)_S}{4} = -\alpha\,\kappa\,\sin\frac{(\phi_1+\phi_2)_W + (\phi_1+\phi_2)_S}{4} \tag{4.96}$$

(see also [32]).

With $\alpha_{10} = \alpha_{32} = \alpha_1$ and $\alpha_{20} = \alpha_{31} = \alpha_2$, (3.26) is satisfied and the remaining permutability condition (3.25) for the primary DBT reads

$$\alpha_1\big(\mathrm{e}^{\mathrm{i}(\phi_0+\phi_1)_S/2} - \mathrm{e}^{\mathrm{i}(\phi_2+\phi_3)_S/2}\big) = \alpha_2\big(\mathrm{e}^{\mathrm{i}(\phi_0+\phi_2)_S/2} - \mathrm{e}^{\mathrm{i}(\phi_1+\phi_3)_S/2}\big) \tag{4.97}$$

from which we obtain again (4.81).

4.2.4. Infinite Toda lattice. Let M^0 be the set of real functions $z_k(t)$, $k \in \mathbb{Z}$, which are smooth in the variable t. On M^0 we define

$$(\delta z)_k = \dot{z}_k\,\tau + (z_{k+1} - z_k)\,\xi \qquad (\mathrm{d}z)_k = (z_k - z_{k-1})\,\tau + \dot{z}_k\,\xi \tag{4.98}$$

where $\dot{z} = \partial z/\partial t$. Together with these maps, $M = M^0 \otimes \Lambda_2$ is a trivial bicomplex [14]. Dressing d with $A = g^{-1}\mathrm{d}g$ where $g = \mathrm{e}^{q_k}$, this yields a bicomplex for the nonlinear Toda lattice equation

$$\ddot{q}_k = \mathrm{e}^{q_{k-1}-q_k} - \mathrm{e}^{q_k-q_{k+1}}. \tag{4.99}$$

Again, we follow scheme (A) of section 3 to determine a primary DBT. Let $g_1 = \mathrm{e}^{p_k}$, $g_2 = \mathrm{e}^{q_k}$ and $(az)_k = \alpha\,z_{k-1}$ with a constant α. Then

$$\frac{1}{\alpha}\,[\tilde{\delta}(g_2^{-1}a\,g_1)z]_k = (\dot{p}_{k-1} - \dot{q}_k)\,\mathrm{e}^{p_{k-1}-q_k}\,z_{k-1}\,\tau + (\mathrm{e}^{p_k-q_{k+1}} - \mathrm{e}^{p_{k-1}-q_k})\,z_k\,\xi \tag{4.100}$$

$$\left(g_2^{-1}\tilde{\mathrm{d}}(g_2 g_1^{-1})\,g_1\,z\right)_k = (\mathrm{e}^{p_{k-1}-p_k} - \mathrm{e}^{q_{k-1}-q_k})\,z_{k-1}\,\tau + (\dot{q}_k - \dot{p}_k)\,\xi \tag{4.101}$$

so that (3.14) yields

$$\dot{q}_k - \dot{p}_k = \alpha\,(\mathrm{e}^{p_k-q_{k+1}} - \mathrm{e}^{p_{k-1}-q_k}) \qquad \dot{p}_{k-1} - \dot{q}_k = \frac{1}{\alpha}\,(\mathrm{e}^{q_k-p_k} - \mathrm{e}^{q_{k-1}-p_{k-1}}). \tag{4.102}$$

We can absorb α in p_k by a redefinition $p_k \mapsto p_k - \ln|\alpha|$ and choose the sign of t such that the above equations become

$$\dot{q}_k - \dot{p}_k = -\mathrm{e}^{p_k-q_{k+1}} + \mathrm{e}^{p_{k-1}-q_k} \qquad \dot{p}_{k-1} - \dot{q}_k = -\mathrm{e}^{q_k-p_k} + \mathrm{e}^{q_{k-1}-p_{k-1}}. \tag{4.103}$$

This is a well known auto-BT of the Toda lattice. From these equations we obtain immediately

$$(\mathrm{e}^{q_k-p_k})^{\cdot} = -\mathrm{e}^{q_k-q_{k+1}} + \mathrm{e}^{p_{k-1}-p_k} \qquad (\mathrm{e}^{p_{k-1}-q_k})^{\cdot} = -\mathrm{e}^{p_{k-1}-p_k} + \mathrm{e}^{q_{k-1}-q_k}. \tag{4.104}$$

Adding these equations and using the Toda equation yields

$$(\mathrm{e}^{q_k-p_k} + \mathrm{e}^{p_{k-1}-q_k})^{\cdot} = -\mathrm{e}^{q_k-q_{k+1}} + \mathrm{e}^{q_{k-1}-q_k} = \ddot{q}_k \tag{4.105}$$

and, after integration,

$$\dot{q}_k = \mathrm{e}^{q_k-p_k} + \mathrm{e}^{p_{k-1}-q_k} - \gamma_k \tag{4.106}$$

with integration constants γ_k. In a similar way, we obtain

$$\dot{p}_k = \mathrm{e}^{q_k-p_k} + \mathrm{e}^{p_k-q_{k+1}} - \tilde{\gamma}_k \tag{4.107}$$

with integration constants $\tilde{\gamma}_k$. Substituting these expressions into (4.103), we find that $\gamma_k = \tilde{\gamma}_k = \gamma$ is a constant, and thus

$$\dot{q}_k = \mathrm{e}^{q_k-p_k} + \mathrm{e}^{p_{k-1}-q_k} - \gamma \qquad \dot{p}_k = \mathrm{e}^{q_k-p_k} + \mathrm{e}^{p_k-q_{k+1}} - \gamma \tag{4.108}$$

which is another form of the auto-BT of the infinite Toda lattice, with a parameter γ [33].

In terms of $g_i = \mathrm{e}^{q_i}$, $i = 0, 1, 2, 3$, we have

$$(R_{ij}z)_k = \alpha_{ij}\,\mathrm{e}^{-q_{i,k}+q_{j,k-1}}\,z_{k-1} \tag{4.109}$$

and with $\alpha_{10} = \alpha_{32} = \alpha_1$ and $\alpha_{20} = \alpha_{31} = \alpha_2$ the permutability conditions amount to

$$q_{3,k} = -q_{0,k-1} + q_{1,k} + q_{2,k} + \ln\frac{\alpha_2\,\mathrm{e}^{q_{1,k-1}} - \alpha_1\,\mathrm{e}^{q_{2,k-1}}}{\alpha_2\,\mathrm{e}^{q_{1,k}} - \alpha_1\,\mathrm{e}^{q_{2,k}}}. \tag{4.110}$$

4.2.5. Hirota's difference equation. For functions $z_k(u, v)$ of three discrete variables k, u, v, we set $(Kz)_k(u, v) = z_{k-1}(u, v)$, $(Uz)_k(u, v) = z_k(u+1, v)$, $(Vz)_k(u, v) = z_k(u, v+1)$, and define bicomplex maps

$$\delta z = (U-1)z\,\xi + (V-1)z\,\tau \qquad \mathrm{d}z = \kappa_1\,(KU-1)z\,\xi + \kappa_2\,(KV-1)z\,\tau. \tag{4.111}$$

With a dressing similar to that for the Toda lattice we obtain a gauge potential one-form

$$A = \mathrm{e}^{-q}\,\tilde{\mathrm{d}}\mathrm{e}^{q} = \kappa_1\,(\mathrm{e}^{KU(q)-q} - 1)\,KU\,\xi + \kappa_2\,(\mathrm{e}^{KV(q)-q} - 1)\,KV\,\tau \tag{4.112}$$

using the notation $K(q) = KqK^{-1}$. Now $\tilde{\delta}A = 0$ with $k \mapsto k+1$ becomes

$$\kappa_1\big(\mathrm{e}^{q_k(u+1,v+1)-q_{k+1}(u,v+1)} - \mathrm{e}^{q_k(u+1,v)-q_{k+1}(u,v)}\big)$$
$$= \kappa_2\big(\mathrm{e}^{q_k(u+1,v+1)-q_{k+1}(u+1,v)} - \mathrm{e}^{q_k(u,v+1)-q_{k+1}(u,v)}\big) \tag{4.113}$$

which, in an equivalent form, is known as *Hirota's bilinear difference equation* [25, 39, 40]. Following scheme (A), we choose $a = \alpha\,K$ with a constant α, so that

$$R = \alpha\,\mathrm{e}^{-q_2}K\mathrm{e}^{q_1} = \alpha\,\mathrm{e}^{K(q_1)-q_2}\,K. \tag{4.114}$$

Now we obtain from (3.14) the BT

$$\alpha\big(\mathrm{e}^{q_{1,k}(u+1,v)-q_{2,k+1}(u+1,v)} - \mathrm{e}^{q_{1,k}(u,v)-q_{2,k+1}(u,v)}\big)$$
$$= \kappa_1\big(\mathrm{e}^{q_{2,k}(u+1,v)-q_{2,k+1}(u,v)} - \mathrm{e}^{q_{1,k+1}(u+1,v)-q_{1,k}(u,v)}\big) \tag{4.115}$$

$$\alpha\big(\mathrm{e}^{q_{1,k}(u,v+1)-q_{2,k+1}(u,v+1)} - \mathrm{e}^{q_{1,k}(u,v)-q_{2,k+1}(u,v)}\big)$$
$$= \kappa_2\big(\mathrm{e}^{q_{2,k}(u,v+1)-q_{2,k+1}(u,v)} - \mathrm{e}^{q_{1,k+1}(u,v+1)-q_{1,k}(u,v)}\big). \tag{4.116}$$

As a 'permutability theorem', we obtain the same formula for $q_{3,k}$ as in the Toda lattice example.

4.2.6. Principal chiral model. Let $M = C^\infty(\mathbb{R}^2, \mathbb{C}^m) \otimes \Lambda_2$ and

$$\delta z = z_t\,\tau + z_x\,\xi \qquad \mathrm{d}z = z_t\,\tau - z_x\,\xi \tag{4.117}$$

for $z \in M^0$. Let G be a group of $m \times m$ matrices. Dressing d with the gauge potential one-form $A = g^{-1}\tilde{\mathrm{d}}g$, where $g \in G$, this yields a bicomplex for the principal chiral model field equation

$$(g^{-1}g_x)_t + (g^{-1}g_t)_x = 0 \tag{4.118}$$

which is $\tilde{\delta}A = 0$ (see also [14]). Hence, we follow scheme (A). With

$$g_2^{-1}\tilde{\mathrm{d}}(g_2 g_1^{-1})\,g_1 = (g_2^{-1}g_{2t} - g_1^{-1}g_{1t})\,\tau - (g_2^{-1}g_{2x} - g_1^{-1}g_{1x})\,\xi \tag{4.119}$$

$$\delta(g_2^{-1}ag_1) = (g_2^{-1}ag_1)_t\,\tau + (g_2^{-1}ag_1)_x\,\xi \tag{4.120}$$

the primary DBT condition (3.14) and $\tilde{\mathrm{d}}a = 0$ requires a to be a constant matrix. Now (3.14) results in the following two equations:

$$g_2^{-1}g_{2t} - g_1^{-1}g_{1t} = (g_2^{-1}a\,g_1)_t \qquad g_2^{-1}g_{2x} - g_1^{-1}g_{1x} = -(g_2^{-1}a\,g_1)_x. \tag{4.121}$$

If a is invertible, the transformation $g_2 \mapsto a\,g_2$ leaves (4.118) invariant and eliminates a from the last equations. This is no longer possible if g is constrained to some subgroup $G \subset GL(m, \mathbb{C})$. For $G = U(m)$, the last equations read

$$g_2^\dagger\,g_{2t} - g_1^\dagger\,g_{1t} = \big(g_2^\dagger a\,g_1\big)_t \qquad g_2^\dagger\,g_{2x} - g_1^\dagger\,g_{1x} = -\big(g_2^\dagger a\,g_1\big)_x. \tag{4.122}$$

Adding the Hermitian conjugates, the resulting equations can be integrated. Using $g_i^\dagger\,g_i = I$, they lead to

$$g_2^\dagger\,a\,g_1 + g_1^\dagger\,a^\dagger\,g_2 = C \tag{4.123}$$

with a constant real matrix C. Assuming again that a is invertible, it can be written as a product uh of a unitary matrix u with a Hermitian matrix h. A redefinition $g_2 \mapsto u\,g_2$ (which leaves the field equations and the unitarity constraint invariant) then eliminates u from the above equations. Hence we can assume that a is Hermitian. Then we find $[g_2^\dagger\, a\, g_1, C] = 0$ (for all g_1, g_2) and thus $C = c\, I$ with $c \in \mathbb{R}$. For $a = \alpha\, I$ with $\alpha \in \mathbb{R}$, we now recover from (4.122) and (4.123) a well known auto-BT for the unitary principal chiral model [34, 35].

With $R_{ij} = g_i^{-1}\, a_{ij}\, g_j$, the permutability conditions take the following form:

$$g_3 = (a_{31}\, g_1 - a_{32}\, g_2)\, g_0^{-1}\, (g_2^{-1}\, a_{20} - g_1^{-1}\, a_{10})^{-1} \qquad a_{31}\, a_{10} = a_{32}\, a_{20}. \tag{4.124}$$

4.2.7. Nonlinear Schrödinger equation. Let $M = C^\infty(\mathbb{R}^2, \mathbb{C}^2) \otimes \Lambda_2$ with

$$\delta z = z_x\, \tau + \tfrac{1}{2}\mathrm{i}(\sigma_3 - I)\, z\, \xi \qquad \mathrm{d}z = z_t\, \tau + z_x\, \xi \tag{4.125}$$

for $z \in M^0$. Furthermore, let $A = -V\, \tau - U\, \xi$ with

$$U = \begin{pmatrix} 0 & -\bar\psi \\ \psi & 0 \end{pmatrix} \qquad V = \mathrm{i}\,(U_x + U^2)\,\sigma_3 = \mathrm{i}\begin{pmatrix} -|\psi|^2 & \bar\psi_x \\ \psi_x & |\psi|^2 \end{pmatrix}. \tag{4.126}$$

This dressing for d yields a bicomplex for the nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0 \tag{4.127}$$

(see also [14]). It is helpful to note that $\sigma_3\, U + U\, \sigma_3 = 0$ and $U^\dagger = -U$. The primary DBT conditions (3.13) now imply the following four equations:

$$R_x = V_1 - V_2 \tag{4.128}$$

$$\tfrac{1}{2}\mathrm{i}\,[\sigma_3, R] = U_1 - U_2 \tag{4.129}$$

$$R_t = V_2\, R - R\, V_1 \tag{4.130}$$

$$R_x = U_2\, R - R\, U_1 \tag{4.131}$$

where U_i, V_i are U, V with ψ replaced by ψ_i, $i = 1, 2$. Decomposing $R = R^+ + R^-$ such that $\sigma_3\, R^\pm = \pm R^\pm\, \sigma_3$, we obtain $R^+ = k\, I + \mathrm{i}\, r\, \sigma_3$ with functions k and r. Furthermore, (4.129) implies

$$R^- = \mathrm{i}\,(U_1 - U_2)\,\sigma_3. \tag{4.132}$$

From (4.128) we obtain $k_x = 0$ and

$$R^+_x = \mathrm{i}\,(|\psi_2|^2 - |\psi_1|^2)\,\sigma_3. \tag{4.133}$$

Hence

$$R = \tilde R + k\, I \qquad \tilde R = \mathrm{i}\begin{pmatrix} r & \bar\psi_1 - \bar\psi_2 \\ \psi_1 - \psi_2 & -r \end{pmatrix} \tag{4.134}$$

with

$$r_x = |\psi_2|^2 - |\psi_1|^2. \tag{4.135}$$

From (4.131) we obtain $\bar k = k$, $\bar r = r$ and

$$(\psi_1 - \psi_2)_x = r\,(\psi_1 + \psi_2) + \mathrm{i}\, k\,(\psi_1 - \psi_2). \tag{4.136}$$

With the help of (4.135), this leads to

$$(|\psi_2 - \psi_1|^2)_x = 2r\,(|\psi_1|^2 - |\psi_2|^2) = -2r r_x \tag{4.137}$$

and, after integration,

$$r = \pm\sqrt{\alpha^2 - |\psi_2 - \psi_1|^2} \tag{4.138}$$

with an integration 'constant' $\alpha^2(t)$. As a consequence, (4.136) becomes

$$(\psi_1 - \psi_2)_x = \pm(\psi_1 + \psi_2)\sqrt{\alpha^2 - |\psi_2 - \psi_1|^2} + \mathrm{i}\, k\, (\psi_1 - \psi_2). \tag{4.139}$$

From (4.130) we obtain $k_t = 0$, so k has to be a constant, and

$$\begin{aligned}(\psi_1 - \psi_2)_t = {} & \pm\mathrm{i}\, (\psi_1 + \psi_2)_x\sqrt{\alpha^2 - |\psi_2 - \psi_1|^2} + \tfrac{1}{2}\mathrm{i}(\psi_1 - \psi_2)\,(|\psi_1 + \psi_2|^2 + |\psi_2 - \psi_1|^2) \\ & - k\,(\psi_1 - \psi_2)_x.\end{aligned} \tag{4.140}$$

The last two equations constitute a well known auto-BT of the nonlinear Schrödinger equation [3, 27, 36–38][32].

Let us turn to the permutability conditions. First we note that $\tilde{R}_{ij}$ is anti-Hermitian, traceless and satisfies

$$\tilde{R}_{ij}\,\tilde{R}_{kl} + \tilde{R}_{kl}\,\tilde{R}_{ij} = [(r_{ij} - r_{kl})^2 + |\psi_i - \psi_j - \psi_k + \psi_l|^2 - \alpha_{ij}^2 - \alpha_{kl}^2]\, I. \tag{4.141}$$

In particular, we have $\tilde{R}_{ij}^2 = -\alpha_{ij}^2\, I$. Using $\mathrm{trace}(\tilde{R}_{ij}) = 0$, (3.24) splits into

$$\tilde{R}_{32} = \tilde{R}_{31} + \tilde{R}_{10} - \tilde{R}_{20} \tag{4.142}$$

$$k_{32} = k_{31} + k_{10} - k_{20}. \tag{4.143}$$

Equation (3.25) reads

$$\tilde{R}_{31}\,\tilde{R}_{10} - \tilde{R}_{32}\,\tilde{R}_{20} + k_{31}\,\tilde{R}_{10} + k_{10}\,\tilde{R}_{31} - k_{32}\,\tilde{R}_{20} - k_{20}\,\tilde{R}_{32} + k_{31}\,k_{10} - k_{32}\,k_{20} = 0 \tag{4.144}$$

and, using (4.142),

$$\begin{aligned}\tilde{R}_{31}(\tilde{R}_{10} - \tilde{R}_{20} + k_{10} - k_{20}) = {} & \tilde{R}_{10}\,\tilde{R}_{20} + (k_{20} - k_{31})\,\tilde{R}_{10} + (k_{32} - k_{20})\,\tilde{R}_{20} \\ & + (\alpha_{20}^2 + k_{32}\,k_{20} - k_{31}\,k_{10})\, I.\end{aligned} \tag{4.145}$$

With the help of

$$(\tilde{R}_{10} - \tilde{R}_{20} + k_{10} - k_{20})^{-1} = -(\tilde{R}_{10} - \tilde{R}_{20} - k_{10} + k_{20})/\rho \tag{4.146}$$

where

$$\rho = |\psi_1 - \psi_2|^2 + (r_{10} - r_{20})^2 + (k_{10} - k_{20})^2 \tag{4.147}$$

this leads to

$$\tilde{R}_{31} = \frac{1}{\rho}\big(a\, I + b\,\tilde{R}_{10} + c\,\tilde{R}_{20} - 2(k_{20} - k_{10})\,\tilde{R}_{10}\,\tilde{R}_{20}\big) \tag{4.148}$$

with

$$\begin{aligned}a = {} & (k_{20} - k_{10})[|\psi_1 - \psi_2|^2 + (r_{10} - r_{20})^2] + (k_{32} - k_{31})\,\alpha_{10}^2 \\ & - (k_{20} - k_{10})(k_{32}\,k_{20} - k_{31}\,k_{10} + \alpha_{20}^2)\end{aligned} \tag{4.149}$$

$$\begin{aligned}b = {} & -|\psi_1 - \psi_2|^2 - (r_{10} - r_{20})^2 + \alpha_{10}^2 - \alpha_{20}^2 + (k_{20} - k_{10})(k_{20} - k_{31}) \\ & - k_{32}\,k_{20} + k_{31}\,k_{10}\end{aligned} \tag{4.150}$$

$$c = \alpha_{20}^2 - \alpha_{10}^2 + (k_{20} - k_{10})(k_{20} - k_{32}) + k_{32}\,k_{20} - k_{31}\,k_{10}. \tag{4.151}$$

Together with $\tilde{R}_{31}^2 = -\alpha_{31}^2\, I$ this implies

$$k_{32} = k_{10} \tag{4.152}$$

and thus $k_{31} = k_{20}$ by use of (4.143), and moreover

$$\alpha_{31}^2 = \alpha_{20}^2. \tag{4.153}$$

[32] The terms proportional to the parameter k are due to the symmetry transformation $\psi(t, x) = \mathrm{e}^{\mathrm{i}\,(-k^2 t + kx)}\,\hat{\psi}(t, x - 2kt)$ of the NLS equation, see also [3], p 68.

Now one can derive from (4.148) the following superposition formula:

$$\psi_3 = \psi_0 + \frac{1}{\rho}\big\{[\alpha_{20}^2 - \alpha_{10}^2 + (k_{20}-k_{10})^2 + 2\mathrm{i}\,(k_{20}-k_{10})\,r_{10}](\psi_2-\psi_0) \\ +[\alpha_{10}^2 - \alpha_{20}^2 + (k_{10}-k_{20})^2 + 2\mathrm{i}\,(k_{10}-k_{20})\,r_{20}](\psi_1-\psi_0)\big\}. \tag{4.154}$$

The remaining equations also require $\alpha_{32}^2 + \alpha_{20}^2 = \alpha_{31}^2 + \alpha_{10}^2$ and thus $\alpha_{32}^2 = \alpha_{10}^2$. With the relations between the parameters of the auto-BT derived above, we recover the permutability theorem as formulated in [38].

Starting with the trivial NLS solution $\psi_0 = 0$, the BT (4.139), (4.140) determines the one-soliton solution

$$\psi(x,t) = \alpha\,\mathrm{e}^{\mathrm{i}\,[k\,x+(\alpha^2-k^2)\,t+\varphi]}/\cosh[\alpha\,(x-x_0) - 2\alpha k t] \tag{4.155}$$

with real constants φ and x_0. Then $r = \alpha\,\tanh[\alpha\,(x-x_0) - 2\,\alpha\,k\,t]$. Let ψ_j, $j = 1, 2$, be two such solutions with parameters $\alpha_j = \alpha_{j0}$, $k_j = k_{j0}$, φ_j and x_j (replacing x_0). Then the above formula for ψ_3 determines a two-soliton solution of the NLS equation.

4.2.8. Discrete nonlinear Schrödinger equation. Let M^0 be the set of $\mathbb{C}^2$-valued functions $z_k(t)$, $k \in \mathbb{Z}$, which are smooth in the time variable t. On M^0 we define

$$(\delta z)_k = (z_{k+1} - z_k)\,\tau - \frac{\mathrm{i}}{2}(\sigma_3 - I)\,z_k\,\xi \qquad (\mathrm{d}z)_k = \dot{z}_k\,\tau + (z_k - z_{k-1})\,\xi \tag{4.156}$$

where a dot denotes a time derivative. This determines a trivial bicomplex. Now we dress d to

$$(\mathrm{D}z)_k = (\dot{z}_k - V_k z_k)\,\tau + (z_k - z_{k-1} - U_k z_{k-1})\,\xi. \tag{4.157}$$

The bicomplex conditions for δ and D are then equivalent to

$$U_{k+1} - U_k - \frac{\mathrm{i}}{2}\,[\sigma_3, V_k] = 0 \tag{4.158}$$

$$\dot{U}_k - (V_k - V_{k-1}) + U_k V_{k-1} - V_k U_k = 0. \tag{4.159}$$

With the decomposition $V_k = V_k^+ + V_k^-$ such that $\sigma_3 V_k^\pm = \pm V_k^\pm \sigma_3$, (4.158) implies

$$V_k^- = \mathrm{i}\,(U_{k+1} - U_k)\,\sigma_3. \tag{4.160}$$

In the following we assume that $\sigma_3\,U_k = -U_k\,\sigma_3$. Now we decompose (4.159) into $\pm$ parts and find

$$V_k^+ = \mathrm{i}\,U_{k+1} U_k\,\sigma_3 \qquad \mathrm{i}\,\dot{U}_k\,\sigma_3 + (U_{k+1} + U_{k-1} - 2U_k) - ({U_k}^2\,U_{k-1} + U_{k+1}\,{U_k}^2) = 0. \tag{4.161}$$

Assuming furthermore ${U_k}^\dagger = -U_k$, we can write

$$U_k = \begin{pmatrix} 0 & -\bar{\psi}_k \\ \psi_k & 0 \end{pmatrix} \tag{4.162}$$

so that

$$V_k = \mathrm{i}\begin{pmatrix} -\bar{\psi}_{k+1}\psi_k & \bar{\psi}_{k+1} - \bar{\psi}_k \\ \psi_{k+1} - \psi_k & \psi_{k+1}\bar{\psi}_k \end{pmatrix} \tag{4.163}$$

and from the second of equations (4.161) we obtain

$$\mathrm{i}\,\dot{\psi}_k + (\psi_{k+1} + \psi_{k-1} - 2\psi_k) - |\psi_k|^2(\psi_{k+1} + \psi_{k-1}) = 0 \tag{4.164}$$

which is the *discrete nonlinear Schrödinger equation* of Ablowitz and Ladik [41]. The equations (3.13), which determine a primary DBT, with $(Rz)_k = P_k z_{k-1}$ lead to

$$P_{k+1} - P_k = -V_{2,k} + V_{1,k} \tag{4.165}$$

$$\frac{\mathrm{i}}{2}\,[\sigma_3, P_k] = -U_{2,k} + U_{1,k} \tag{4.166}$$

$$\dot{P}_k = V_{2,k}\,P_k - P_k\,V_{1,k-1} \tag{4.167}$$

$$P_k - P_{k-1} = U_{2,k}\,P_{k-1} - P_k\,U_{1,k-1}. \tag{4.168}$$

Decomposing $P_k = P_k^+ + P_k^-$, (4.166) implies

$$P_k^- = \mathrm{i}\,(U_{1,k} - U_{2,k})\,\sigma_3 \tag{4.169}$$

and from (4.165) we obtain

$$P_{k+1}^+ - P_k^+ = \mathrm{i}\,(U_{1,k+1}\,U_{1,k} - U_{2,k+1}\,U_{2,k})\,\sigma_3. \tag{4.170}$$

Setting

$$P_k^+ = \mathrm{i}\begin{pmatrix} p_k & 0 \\ 0 & \bar{p}_k \end{pmatrix}\sigma_3 \tag{4.171}$$

we have

$$P_k = \mathrm{i}\begin{pmatrix} p_k & \bar{\psi}_{1,k} - \bar{\psi}_{2,k} \\ \psi_{1,k} - \psi_{2,k} & -\bar{p}_k \end{pmatrix} \tag{4.172}$$

and (4.170) becomes

$$p_{k+1} - p_k = \bar{\psi}_{2,k+1}\,\psi_{2,k} - \bar{\psi}_{1,k+1}\,\psi_{1,k}. \tag{4.173}$$

Now we obtain from (4.168)

$$(\psi_1 - \psi_2)_k - (\psi_1 - \psi_2)_{k-1} = p_{k-1}\,\psi_{2,k} + \bar{p}_k\,\psi_{1,k-1} \tag{4.174}$$

and (4.167) leads to

$$\begin{aligned}(\psi_{1,k} - \psi_{2,k})^{\dot{}} &= \mathrm{i}\,(\psi_{1,k} - \psi_{2,k})(\bar{\psi}_{1,k}\psi_{1,k-1} + \bar{\psi}_{2,k}\psi_{2,k+1}) \\ &\quad + \mathrm{i}\,p_k\,(\psi_{2,k+1} - \psi_{2,k}) + \mathrm{i}\,\bar{p}_k\,(\psi_{1,k} - \psi_{1,k-1}).\end{aligned} \tag{4.175}$$

In order to obtain a BT one has to eliminate p_k from (4.174) and (4.175) with the help of (4.173). However, there seems to be no convenient way to achieve this.

4.2.9. A generalized Volterra equation. Let M^0 be the set of functions $z_n(t)$, $n \in \mathbb{Z}$, which are smooth in the variable t and which have values in $\mathbb{C}^m$, $m \in \mathbb{N}$. On M^0 we define

$$(\delta z)_n = (z_n - z_{n-k})\,\xi + \dot{z}_n\,\tau \qquad (\mathrm{d}z)_n = (z_{n+1} - z_n)\,\xi + (z_{n+k+1} - z_n)\,\tau \tag{4.176}$$

for some fixed $k \in \mathbb{Z}$. Then $M = M^0 \otimes \Lambda_2$ together with δ and d is a trivial bicomplex. Now we introduce a dressing:

$$(\mathrm{D}z)_n = (g^{-1}\,\mathrm{d}\,gz)_n = (g_n^{-1}g_{n+1}z_{n+1} - z_n)\,\xi + (g_n^{-1}g_{n+k+1}z_{n+k+1} - z_n)\,\tau \tag{4.177}$$

with an invertible $m \times m$ matrix g, depending on t and the discrete variable n. Introducing the abbreviation

$$V_n = g_n^{-1}\,g_{n+1} \tag{4.178}$$

the operator D can be expressed as follows:

$$(\mathrm{D}z)_n = (V_n\,z_{n+1} - z_n)\,\xi + (V_n V_{n+1}\ldots V_{n+k}\,z_{n+k+1} - z_n)\,\tau. \tag{4.179}$$

The only nontrivial bicomplex condition is $\delta\mathrm{D} + \mathrm{D}\delta = 0$. It results in the generalized Volterra equation

$$\dot{V}_n = V_n V_{n+1} \ldots V_{n+k} - V_{n-k} V_{n-k+1} \ldots V_n \tag{4.180}$$

which is also known as one of the Bogoyavlenskiĭ lattices ([42], see also [43]). For $k = 1$ it reduces to (a matrix version of) the Volterra equation

$$\dot{V}_n = V_n\, V_{n+1} - V_{n-1}\, V_n. \tag{4.181}$$

In order to elaborate primary DBTs, we have to solve the equations (3.13). With the ansatz $(Rz)_n = r_n\, z_{n+k+1}$, we obtain the equations

$$r_n - r_{n-k} = V_{2,n} - V_{1,n} \qquad \dot{r}_n = V_{2,n} \ldots V_{2,n+k} - V_{1,n} \ldots V_{1,n+k} \tag{4.182}$$

and

$$V_{2,n}\, r_{n+1} = r_n\, V_{1,n+k+1}. \tag{4.183}$$

Expressing V back in terms of g, we can 'integrate' the last equation and obtain

$$r_n = g_{2,n}^{-1}\, a\, g_{1,n+k+1} \tag{4.184}$$

with an arbitrary $m \times m$ matrix $a(t)$. Inserted in the two equations (4.182), this leads to the following BT for the generalized Volterra equation:

$$g_{2,n}^{-1}\, g_{2,n+1} - g_{1,n}^{-1}\, g_{1,n+1} = g_{2,n}^{-1}\, a\, g_{1,n+k+1} - g_{2,n-k}^{-1}\, a\, g_{1,n+1} \tag{4.185}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{2,n}^{-1}\, a\, g_{1,n+k+1}\right) = g_{2,n}^{-1}\, g_{2,n+k+1} - g_{1,n}^{-1}\, g_{1,n+k+1}. \tag{4.186}$$

As a permutability relation, we obtain

$$g_{3,n} = \left(a_2\, g_{1,n+k+1} - a_1\, g_{2,n+k+1}\right) g_{0,n+k+1}^{-1} \left(g_{2,n}^{-1}\, a_2 - g_{1,n}^{-1}\, a_1\right)^{-1} \tag{4.187}$$

where $a_{10} = a_{32} = a_1$ and $a_{20} = a_{31} = a_2$ and $[a_1, a_2] = 0$.

Let us now restrict our considerations to the *scalar* case where $m = 1$. Inserting

$$g_n = \frac{f_n}{f_{n-k-1}} \tag{4.188}$$

into the BT part (4.185) yields

$$\frac{f_{1,n}\, f_{2,n+1} - a\, f_{1,n+k+1}\, f_{2,n-k}}{f_{1,n+1}\, f_{2,n}} = \frac{f_{1,n-k-1}\, f_{2,n-k} - a\, f_{1,n}\, f_{2,n-2k-1}}{f_{1,n-k}\, f_{2,n-k-1}} \tag{4.189}$$

which can be 'integrated':

$$f_{1,n}\, f_{2,n+1} - a\, f_{1,n+k+1}\, f_{2,n-k} = \beta_i\, f_{1,n+1}\, f_{2,n} \tag{4.190}$$

where β_i with $n = \mathrm{i} \bmod (k+1)$ are 'constants of integration'. The complementary BT part (4.186) with $\dot{a} = 0$ becomes

$$\frac{a\, D_t(f_{1,n+k+1} \cdot f_{2,n}) - f_{1,n}\, f_{2,n+k+1}}{f_{1,n+k+1}\, f_{2,n}} = \frac{a\, D_t(f_{1,n} \cdot f_{2,n-k-1}) - f_{1,n-k-1}\, f_{2,n}}{f_{1,n}\, f_{2,n-k-1}} \tag{4.191}$$

using Hirota's bilinear operator $D_t(f \cdot h) = \dot{f}h - f\dot{h}$. This can also be integrated with the result

$$a\, D_t(f_{1,n+k+1} \cdot f_{2,n}) - f_{1,n}\, f_{2,n+k+1} = \gamma_i\, f_{1,n+k+1}\, f_{2,n} \tag{4.192}$$

where γ_i with $n = \mathrm{i} \bmod (k+1)$ are 'constants of integration'. Here we have obtained a BT in Hirota's bilinear form.

5. Harry Dym equation and equivalence transformations of bicomplexes

Let $M = C^\infty(\mathbb{R}^2, \mathbb{R}) \otimes \Lambda_2$. With

$$\mathcal{D}z = [\varphi\, z_x - \tfrac{1}{2}\varphi_x\, z]\, \xi + [3\varphi^2\, z_{xx} + \tfrac{3}{4}({\varphi_x}^2 - 2\varphi\, \varphi_{xx})\, z]\, \tau \tag{5.1}$$

$$\mathrm{D}z = \varphi^2\, z_{xx}\, \xi + (z_t + 4\varphi^3\, z_{xxx} + 6\varphi^2\, \varphi_x\, z_{xx})\, \tau \tag{5.2}$$

we obtain $\mathcal{D}^2 = 0$ identically, and

$$\mathrm{D}^2 = \xi\, \tau\, \{-2\, \varphi\, [\varphi_t + \varphi^3\, \varphi_{xxx}]\, \partial_x^2\} \tag{5.3}$$

$$\mathcal{D}\, \mathrm{D} + \mathrm{D}\, \mathcal{D} = \xi\, \tau\, \{\tfrac{1}{2}\, [\varphi_t + \varphi^3\, \varphi_{xxx}]_x - [\varphi_t + \varphi^3\, \varphi_{xxx}]\, \partial_x\} \tag{5.4}$$

so that the bicomplex conditions are equivalent to the HD equation

$$\varphi_t + \varphi^3\, \varphi_{xxx} = 0. \tag{5.5}$$

A relation between the HD and the KdV equation has been the subject of several publications [23,44]. In the following, we show how such a relation emerges in our bicomplex framework. This is an instructive example of the application of equivalence transformations to bicomplexes. With the gauge transformation

$$\mathcal{D}' = \varphi^{-1/2} \mathcal{D}\, \varphi^{1/2} \qquad \mathrm{D}' = \varphi^{-1/2} \mathrm{D}\, \varphi^{1/2} \tag{5.6}$$

(assuming φ to vanish nowhere) we obtain an equivalent bicomplex with

$$\mathcal{D}' z = (\varphi \partial_x) z\, \xi + 3(\varphi \partial_x)^2 z\, \tau \tag{5.7}$$

$$\mathrm{D}' z = [(\varphi\partial_x)^2 + \tfrac{1}{4}(2\varphi\varphi_{xx} - {\varphi_x}^2)]\, z\, \xi + [\partial_t + 4(\varphi\partial_x)^3 + (\varphi\partial_x)(2\varphi\varphi_{xx} - {\varphi_x}^2) + \tfrac{1}{2}\varphi^{-1}\, \varphi_t]\, z\, \tau. \tag{5.8}$$

Next we perform a change of coordinates $x = v(s, y)$, $t = s$ such that $v_y = \varphi$. As a consequence, $\partial_y = v_y\, \partial_x = \varphi\, \partial_x$, $\partial_t = \partial_s + y_t\, \partial_y$, and $0 = x_t = v_s + y_t\, v_y$ so that $y_t = -v_s/v_y$. Writing $z(t, x) = \zeta(s, y)$, the bicomplex maps $\mathcal{D}'$ and D' take the following form in the new coordinates:

$$\mathcal{D}'\zeta = \zeta_y\, \xi + 3\, \zeta_{yy}\, \tau \qquad \mathrm{D}'\zeta = (\zeta_{yy} - u\, \zeta)\, \xi + (\zeta_s + 4\, \zeta_{yyy} - p\, \zeta_y - q\, \zeta)\, \tau \tag{5.9}$$

where

$$p = 4\, u + \frac{v_s}{v_y} \qquad q = (\tfrac{1}{2}\, p + 6\, u)_y \qquad u = -\frac{1}{2} \frac{v_{yyy}}{v_y} + \frac{3}{4} \left(\frac{v_{yy}}{v_y}\right)^2 = -\frac{1}{2}\, \mathcal{S}_y v \tag{5.10}$$

and $\mathcal{S}_y$ denotes the Schwarzian derivative. The bicomplex conditions for the maps (5.9) reduce to

$$p = 6\, u \tag{5.11}$$

where u has to satisfy the KdV equation

$$u_s + u_{yyy} - 6\, u\, u_y = 0. \tag{5.12}$$

Equation (5.11) together with (5.10) yields[33]

$$v_s = -(\mathcal{S}_y v)\, v_y \tag{5.13}$$

which must be equivalent to the HD equation. The KdV equation for u is satisfied as a consequence of this equation.

In terms of $\psi = 1/\sqrt{v_y}$ the definition of u reads $\psi_{yy} = u\, \psi$. Given an HD solution $\varphi(t, x)$, we have to invert $y(t, x) = \int (1/\varphi)\, \mathrm{d}x$ to determine $x = v(s, y)$. Then $u = \psi_{yy}/\psi$ is a KdV solution. Now we can use a KdV–BT to construct a new solution $\hat{u}$ of the KdV equation.

[33] This is the simplest case of a Krichever–Novikov equation [45].

After solving $\hat{\psi}_{yy} = \hat{u}\,\hat{\psi}$ for $\hat{\psi}$, we have to invert $\hat{v} = \int (1/\hat{\psi}^2)\,\mathrm{d}y$ to find $\hat{y} = y(t,x)$. Then $\hat{\varphi} = 1/\hat{y}_x$ is again a solution of the HD equation. In particular, if χ satisfies the first of equations (4.12), i.e. $\chi_{yy} = (u-\alpha)\,\chi$ with a function $\alpha(s)$, then $\hat{\psi} = \psi_y - (\ln\chi)_y\psi$ (which is a Darboux transformation [10]) satisfies $\hat{\psi}_{yy} = \hat{u}\,\hat{\psi}$ with $\hat{u} = u - 2(\ln\chi)_{yy}$. If χ also satisfies the second of equations (4.12) (with t, x replaced by s, y), then $\hat{u}$ is a KdV solution and from $\hat{\psi}$ we obtain an HD solution.

As an example, let us start with the trivial solution $\varphi = 1$ of the HD equation. Then we obtain $y = x + a(t)$ and thus $v = y - a(s)$ with an 'integration constant' a. Then we have $\psi = 1$ and consequently $u = 0$, which trivially solves the KdV equation. Furthermore, the equation $\chi_{yy} = (u + k^2)\chi$ with a constant k and also the second of equations (4.12) is solved by $\chi = \chi_0\cosh(ky - 4k^3s)$. Now we find $\hat{\psi} = -k\tanh(ky - 4k^3s)$ and $\hat{v} = y/k^2 - \coth(ky - 4k^3s)/k^3$. $\hat{u} = -2k^2\,\mathrm{sech}^2(ky - 4k^3s)$ is the one-soliton KdV solution. In order to obtain an HD solution, we have to solve $x = \hat{v}(s,y)$ for y, which results in a function $\hat{y}(t,x)$ with $t = s$. This cannot be done explicitly, but we find $\hat{\varphi} = k^{-2}[1 + 1/\sinh^2(4k^3t - k\hat{y}(t,x))]$, which indeed solves the HD equation.

Of course, one can try to solve the auto-DBT condition for the bicomplex $(M, \mathcal{D}, \mathrm{D})$ associated with the HD equation, using (1.15). This turns out to be rather difficult since already the solution for $Q^{(0)}$ is a nonpolynomial differential operator. It appears to be more convenient to work with the equivalent bicomplex $(M, \mathcal{D}', \mathrm{D}')$. However, the latter is tied to a less convenient form of the HD equation.

6. Conclusions

We have introduced the concept of a DBT of a bicomplex and demonstrated in several examples how BTs for integrable models are easily obtained using this simple and universal construction. Once a bicomplex formulation is found for some equation, it is straightforward, in general, to apply this method. The bicomplex structure does not guarantee a 'decent' BT, however. In some cases, the resulting correspondence between solutions appears to be practically not of much help (cf the example of the discrete NLS equation in section 4.2).

Higher than primary DBTs have not been sufficiently elaborated in this work, with the exception of the Liouville example in section 2. In the KdV case, the secondary DBT turned out to be a composition of two primary DBTs. More precisely, for one choice of sign of a real parameter this can only be achieved if one generalizes the primary DBTs to include complex transformations. Although the latter do not, in general, generate real solutions from real solutions, their composition does. Hence, if we reduce our framework to real solutions and real maps, *not* all of the secondary DBTs are compositions of primary DBTs. Corresponding results certainly also hold for higher than secondary DBTs in the KdV case. This shows that, in general, we should not expect compositions of primary DBTs to exhaust the hierarchy of DBTs.

Suppose we have three equations EQ_i, $i = 1, 2, 3$, which are reductions of equations $\widehat{EQ}_i$ with bicomplex formulations. If u_i is a solution of EQ_i, then u_i is also a solution of $\widehat{EQ}_i$. Let $\mathcal{S}_i$ and $\hat{\mathcal{S}}_i$ denote the solution spaces of EQ_i and $\widehat{EQ}_i$, respectively. Suppose there are BTs $\widehat{BT}_{21} : \hat{\mathcal{S}}_1 \to \hat{\mathcal{S}}_2$ and $\widehat{BT}_{32} : \hat{\mathcal{S}}_2 \to \hat{\mathcal{S}}_3$ (see figure 2) which are determined by primary DBTs. Let $u_1 \in \mathcal{S}_1$ and thus also $u_1 \in \hat{\mathcal{S}}_1$. Applying $\widehat{BT}_{32}\widehat{BT}_{21}$ to it yields some $\hat{u}_3 \in \hat{\mathcal{S}}_3$ which can be projected to some $u_3 \in \mathcal{S}_3$. Not all of such composed and then reduced maps will be trivial and not all of them should be expected to be obtainable from primary DBTs of the reduced bicomplexes. However, such maps may be recovered as higher than primary DBTs between the initial and the final reduced bicomplex. A very simple example is indeed

$$\begin{tikzcd}
\hat{\mathcal{S}}_1 \arrow[r, "\widehat{BT}_{21}"] \arrow[d, "\pi_1"'] & \hat{\mathcal{S}}_2 \arrow[r, "\widehat{BT}_{32}"] \arrow[d, "\pi_2"'] & \hat{\mathcal{S}}_3 \arrow[d, "\pi_3"'] \\
\mathcal{S}_1 \arrow[r, dashed] & \mathcal{S}_2 \arrow[r, dashed] & \mathcal{S}_3
\end{tikzcd}$$

Figure 2. BTs and reductions (the maps π_i).

provided by the KdV equation mentioned above. In this case, all three equations EQ_i are given by the *real* KdV equation and $\widehat{EQ}_i$ is the complex KdV equation (where the dependent variable has values in $\mathbb{C}$).

The method presented in this work is a constructive one, a receipe to determine BTs. We do not know how exhaustive it is and the techniques developed here do not provide us with suitable tools to answer this question. Other techniques are available, of course, like those in the jet-bundle framework (see [3], chapter 2, and [46], for example).

In our examples, we have concentrated on equations in two (continuous or discrete) dimensions, with the exception of Hirota's difference equation, which depends on three discrete variables. Of course, the method also applies to other higher-dimensional equations possessing a bicomplex formulation (cf [14, 15]). More examples of this kind will be studied elsewhere.

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